Evaluating multiloop Feynman integrals
by differential equations

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Historiographical summary
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General prescriptions and a simple one-loop example
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Massless three-loop four-point Feynman integrals on the light cone
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Conclusion
Gehrmann & Remiddi: a method to evaluate *master integrals*. It is assumed that the problem of reduction to master integrals is solved.
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Henn: use uniform transcendentality (UT)!
Reduction to master integrals

Evaluating a family of Feynman integrals associated with a given graph with general integer powers of the propagators (indices)

\[ F_\Gamma(q_1, \ldots, q_n; d; a_1, \ldots, a_L) = \int \ldots \int I(q_1, \ldots, q_n; k_1, \ldots, k_h; a_1, \ldots, a_L) \, d^d k_1 \, d^d k_2 \ldots \, d^d k_h \]

\[ I(q_1, \ldots, q_n; k_1, \ldots, k_h; a_1, \ldots, a_L) = \frac{1}{(p_1^2 - m_1^2)^{a_1} (p_2^2 - m_2^2)^{a_2} \ldots} \]
The old straightforward analytical strategy:

to evaluate, by some methods, every scalar Feynman integral generated by the given graph.
The **standard** modern strategy:

to derive, without calculation, and then apply IBP identities between the given family of Feynman integrals as recurrence relations.
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The whole problem of evaluation →

- constructing a reduction procedure
- evaluating master integrals
General prescriptions
Take some derivatives of given master integrals in masses or/and kinematic invariants
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- Express them in terms of Feynman integrals of the given family with shifted indices
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- Take some derivatives of given master integrals in masses or/and kinematic invariants
- Express them in terms of Feynman integrals of the given family with shifted indices
- Apply an IBP reduction (using some public or private code) to express these integrals in terms of master integrals to obtain a system of differential equations
- Solve DE
The crucial point:

choose all the master integrals as pure functions of uniform weight, i.e.
uniform degree of transcendentality
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Weight for numbers: $n$ for $\zeta(n)$, $\text{Li}_n(1/2)$ etc.
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A transition to a UT basis is a linear transformation in the space of master integrals and the corresponding matrix is rational with respect to dimension and kinematic invariants.
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A decisive criterion: if we arrive at canonical DE then we make a proper choice of UT master integrals!
An example: a one-loop massless propagator integral

\[
\int \frac{d^d k}{(-k^2)^{a_1}[-(q-k)^2]^{a_2}} = i \pi^{d/2} \frac{G(a_1, a_2)}{(-q^2)^{a_1+a_2+\epsilon-2}},
\]

\[
G(a_1, a_2) = \frac{\Gamma(a_1 + a_2 + \epsilon - 2)\Gamma(2 - \epsilon - a_1)\Gamma(2 - \epsilon - a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(4 - a_1 - a_2 - 2\epsilon)}
\]

with \( d = 4 - 2\epsilon \)
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\]

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\( \Gamma(1 + k\epsilon), \Gamma(k\epsilon) \) are UT, e.g.

\[
\Gamma(1 + \epsilon) = e^{-\gamma_E \epsilon} \left( 1 + \frac{\pi^2 \epsilon^2}{12} - \frac{\epsilon^3 \zeta(3)}{3} + \ldots \right)
\]

\( \Gamma(2 - 2\epsilon) \equiv (1 - 2\epsilon)\Gamma(1 - 2\epsilon) \) is not UT
\[ G(1, 1) = \frac{\Gamma(1-\epsilon)^2 \Gamma(\epsilon)}{\Gamma(2-2\epsilon)} \text{ is not UT} \]
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\[ G(2, 1) = \frac{\Gamma(1-\epsilon) \Gamma(-\epsilon) \Gamma(\epsilon+1)}{\Gamma(1-2\epsilon)} \text{ is UT} \]
One can use Feynman parameters. For example,

\[
\int \frac{d^d k}{(-k^2 + m^2)^{a_1} [-(q - k)^2]^{a_2}} \sim \int_0^1 \frac{\alpha^{a_2 - 1} (1 - \alpha)^{1-a_2-\epsilon}}{[1 + x\alpha]^{a_1+a_2+\epsilon-2}}
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\]

A general rule: factors like \((1 - \alpha)^{\pm \epsilon}\) or \(\alpha^{\pm \epsilon}\) do not spoil UT
Replace propagators by delta functions. An example: the on-shell box with $p_i^2 = 0$ and $s = (p_1 + p_2)^2$ and $t = (p_1 + p_3)^2$. 
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$$\int \frac{d^d k}{k^2 (k + p_1)^2 (k + p_1 + p_2)^2 (k - p_3)^2} \to$$

$$\int d^4 k \delta(k^2) \delta((k + p_1)^2) \delta((k + p_1 + p_2)^2) \delta((k - p_3)^2) \sim \frac{1}{st}$$
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This gives the hint that after the multiplication by $st$ we should obtain a UT Feynman integral.
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\[
\partial_i f(\epsilon, x) = A_i(\epsilon, x) f(\epsilon, x),
\]

where \( \partial_i = \frac{\partial}{\partial x_i} \), and each \( A_i \) is an \( N \times N \) matrix.
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\]

How to prove it? (A good mathematical problem.)
An example: the massless on-shell box diagram, i.e. with
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$$F_{\Gamma}(s, t; a_1, a_2, a_3, a_4, d) = \int \frac{d^d k}{(-k^2)^{a_1}[-(k + p_1)^2]^{a_2}[-(k + p_1 + p_2)^2]^{a_3}[-(k - p_3)^2]^{a_4}}$$,

where $s = (p_1 + p_2)^2$ and $t = (p_1 + p_3)^2$.
Three master integrals $F(0, 1, 0, 1), F(1, 0, 1, 0), F(1, 1, 1, 1)$. 
Three master integrals $F(0, 1, 0, 1), F(1, 0, 1, 0), F(1, 1, 1, 1)$. The first two of them are given in terms of gamma functions. Choose them proportional to $G(2, 1) = \frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(\epsilon+1)}{\Gamma(1-2\epsilon)}$
Three master integrals $F(0, 1, 0, 1), F(1, 0, 1, 0), F(1, 1, 1, 1)$. The first two of them are given in terms of gamma functions. Choose them proportional to $G(2, 1) = \frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(\epsilon+1)}{\Gamma(1-2\epsilon)}$

Turn to a UT basis:

$$f = (-s)^\epsilon \{ \epsilon t F(0, 1, 0, 2), \epsilon s F(1, 0, 2, 0), \epsilon^2 st F(1, 1, 1, 1) \}$$

$$\equiv \{ f_1, f_2, f_3 \}$$

with $x = t/s, s = -1$. 
DE in the new basis

\[ f' (\epsilon, x) = \epsilon A(x) f (\epsilon, x) \]
DE in the new basis

\[ f'(\epsilon, x) = \epsilon A(x) f(\epsilon, x) \]

where

\[
A(x) = \begin{pmatrix}
-\frac{1}{x} & 0 & 0 \\
0 & 0 & 0 \\
\frac{2}{x+1} - \frac{2}{x} & \frac{2}{x+1} & \frac{1}{x+1} - \frac{1}{x}
\end{pmatrix}
\]
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\[ A(x) = \begin{pmatrix} -\frac{1}{x} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2}{x+1} - \frac{2}{x} & \frac{2}{x+1} & \frac{1}{x+1} - \frac{1}{x} \end{pmatrix} \]

Solving DE in the \( \epsilon \)-expansion, \( f = \sum_{n=0} f^{(n)} \epsilon^n \)

\[ \frac{d}{dx} f^{(n)}(x) = A(x) f^{(n-1)}(x) . \]
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\[
\frac{d}{dx} f^{(n)}(x) = A(x) f^{(n-1)}(x).
\]

\[
f^{(n)}(x) = \int_0^x dx' A(x') f^{(n-1)}(x') + g^{(n)}.
\]
The boundary conditions $g^{(n)}$ are fixed at the point $x = -1$ (i.e. $s + t \equiv -u = 0$) where the given integral is not singular.
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In every order of the $\epsilon$-expansion, one obtains a linear combination of integrals

$$\int_{0 \leq x_1 \leq \ldots \leq x_k \leq x} \frac{dx_k}{x_k + a_k} \ldots \frac{dx_1}{x_1 + a_1}$$

where $a_i = 0$ or $1$. 
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\]

where \( a_i = 0 \) or 1.

HPLs

\[
H(a_1, a_2, \ldots, a_n; x) = \int_0^x f(a_1; t)H(a_2, \ldots, a_n; t)dt,
\]

where \( f(\pm1; t) = 1/(1 \mp t), \ f(0; t) = 1/t \)
The result is $f_3 = \sum_{j=0} c_j(x, L) e^j$, with

$$
c_0 = 4, \quad c_1 = 2L, \quad c_2 = -\frac{4}{3} \pi^2,
$$

$$
c_3 = \pi^2 H_1(x) + 2H_{0,0,1}(x) - \frac{7}{6} \pi^2 L + 2H_{0,1}(x) L + H_1(x) L^2 - \frac{1}{3} L^3 - \frac{34}{3} \zeta_3,
$$

$$
c_4 = -2H_{1,0,0,1}(x) - 2H_{0,0,1,1}(x) - 2H_{0,1,0,1}(x) - 2H_{0,0,0,1}(x) - 2H_{0,1,1}(x) L
- 2H_{1,0,1}(x) L + H_{0,1}(x) L^2 - H_{1,1}(x) L^2 + \frac{2}{3} H_1(x) L^3 - \frac{1}{6} L^4
- \pi^2 H_{1,1}(x) + \pi^2 H_1(x) L - \frac{1}{2} \pi^2 L^2 + 2H_1(x) \zeta_3 - \frac{20}{3} L \zeta_3 - \frac{41}{360} \pi^4 + \cdots,
$$

with $L = \log x$. 
Massless three-loop four-point Feynman integrals on the light cone

(A)

(E)
\[ F^A_{a_1, \ldots, a_{15}}(s, t; D) = \int \int \int \frac{d^D k_1 \, d^D k_2 \, d^D k_3}{(-k_1^2)^{a_1} \, [-(p_1 + p_2 + k_1)^2]^{a_2} \, (-k_2^2)^{a_3}} \times \]
\[ \left[ -(k_1 - p_3)^2 \right]^{-a_{11}} \, \left[ -(p_1 + k_2)^2 \right]^{-a_{12}} \, \left[ -(k_2 - p_3)^2 \right]^{-a_{13}} \times \]
\[ \left[ -(p_1 + p_2 + k_2)^2 \right]^{a_4} \, \left[ -k_3^2 \right]^{a_5} \, \left[ -(p_1 + p_2 + k_3)^2 \right]^{a_6} \, \left[ -(p_1 + k_1)^2 \right]^{a_7} \]
\[ \times \left[ -(k_1 - k_2)^2 \right]^{a_8} \, \left[ -(k_2 - k_3)^2 \right]^{a_9} \, \left[ -(k_3 - p_3)^2 \right]^{a_{10}} \].\]

\[ F^E_{a_1, \ldots, a_{15}}(s, t; D) = \int \int \int \frac{d^D k_1 \, d^D k_2 \, d^D k_3}{[-(k_1 - k_3)^2]^{a_1} \, [-(p_1 + k_1)^2]^{a_2}} \times \]
\[ \left[ -(p_1 + p_2 + k_3)^2 \right]^{-a_{11}} \, \left[ -(p_1 + k_2)^2 \right]^{-a_{12}} \, \left[ -(k_1 - p_3)^2 \right]^{-a_{13}} \times \]
\[ \left[ -(p_1 + p_2 + k_1)^2 \right]^{a_3} \, \left[ -(p_1 + p_2 + k_2)^2 \right]^{a_4} \, \left[ -k_2^2 \right]^{a_5} \, \left[ -(k_2 - p_3)^2 \right]^{a_6} \]
\[ \times \left[ -k_1^2 \right]^{-a_{14}} \, \left[ -k_2^2 \right]^{-a_{15}} \times \]
\[ \left[ -(k_1 - k_2)^2 \right]^{a_7} \, \left[ -k_3^2 \right]^{a_8} \, \left[ -(p_1 + k_3)^2 \right]^{a_9} \, \left[ -(k_3 - p_3)^2 \right]^{a_{10}} \].\]
(17) \hspace{1cm} (20) \hspace{1cm} (21) \hspace{1cm} (15) \hspace{1cm} (16) \\
(24), (25)^*, (26)^*
(32)

(33)*

(34)*

(35)*

(36)*

(37)*

(38)*

(39)*,

(40)*, (41)*
\[ f_i^A = \epsilon^3 (-s)^{3\epsilon} \frac{e^{3\epsilon\gamma_E}}{(i\pi D/2)^3} g_i^A. \]

The factor \((-s)^{3\epsilon}\) is to make the basis functions \(f_i^A\) dimensionless.
The factor \(\epsilon^3\) ensures that all basis functions admit a Taylor expansion around \(\epsilon = 0\).
\[ g_1^A = t F_0^A 0,0,0,0,0,2,2,2,1,0,0,0,0,0,0, \quad g_2^A = s F_0^A 0,2,0,0,1,0,2,2,0,0,0,0,0,0,0, \]
\[ g_3^A = \epsilon s F_0^A 0,0,0,0,1,1,2,2,1,0,0,0,0,0,0, \quad g_4^A = \epsilon s F_0^A 0,0,0,1,2,0,2,1,1,0,0,0,0,0,0,0, \]
\[ g_5^A = s F_0^A 0,1,2,-1,0,1,0,2,2,0,0,0,0,0,0, \quad g_6^A = s^2 F_0^A 0,2,2,0,2,1,0,1,0,0,0,0,0,0,0,0, \]
\[ g_7^A = \epsilon s t F_0^A 0,0,0,0,1,1,2,2,1,1,0,0,0,0,0,0, \quad g_8^A = \epsilon^2 (s + t) F_0^A 0,0,0,1,1,0,2,1,1,1,0,0,0,0,0,0, \]
\[ g_9^A = \epsilon s t F_0^A 0,0,1,1,0,0,2,1,1,2,0,0,0,0,0,0, \quad g_{10}^A = \epsilon s^2 F_0^A 0,0,1,1,2,1,2,1,0,0,0,0,0,0,0,0, \]
\[ g_{11}^A = \epsilon^2 (s + t) F_0^A 0,1,0,0,1,0,1,2,1,0,0,0,0,0,0,0, \quad g_{12}^A = -\epsilon (2\epsilon - 1) s F_1^A 0,1,0,0,1,1,0,2,1,0,0,0,0,0,0,0, \]
\[ g_{13}^A = s^3 F_2^A 0,1,2,1,2,1,0,0,0,0,0,0,0,0,0,0, \quad g_{14}^A = \epsilon s F_0^A 0,0,1,1,0,0,2,1,1,2,0,0,-1,0,0, \]
\[ g_{15}^A = \epsilon s^3 t F_0^A 0,1,1,0,0,0,1,1,1,1,0,0,0,0,0,0, \quad g_{16}^A = \epsilon^2 s^2 F_0^A 0,1,2,0,0,1,1,1,1,0,0,0,0,0,0,0, \]
\[ g_{17}^A = \epsilon^3 s F_0^A 0,1,1,0,1,1,1,1,1,0,0,0,0,0,0,0, \quad g_{18}^A = \epsilon^2 s^2 F_0^A 0,0,1,1,1,2,1,1,1,0,0,-1,0,0, \]
\[ g_{19}^A = \epsilon^2 s^2 t F_0^A 0,0,1,1,1,1,2,1,1,1,0,0,0,0,0,0, \quad g_{20}^A = \epsilon^3 s(s + t) F_0^A 0,1,1,0,1,1,1,1,1,0,0,0,0,0,0,0, \]
\[ g_{21}^A = \epsilon^2 s^2 t F_0^A 0,1,1,0,0,1,1,1,2,1,1,0,0,0,0,0, \quad g_{22}^A = \epsilon^2 s^2 t F_1^A 0,1,0,0,1,1,1,2,1,1,0,0,0,0,0,0, \]
\[ g_{23}^A = \epsilon^2 s^2 F_1^A 0,1,0,0,1,1,1,2,1,1,-1,0,0,0,0,0, \quad g_{24}^A = \epsilon^3 s^3 t F_1^A 0,1,1,1,1,1,1,1,1,0,0,0,0,0,0,0, \]
\[ g_{25}^A = \epsilon^3 s^3 F_1^A 0,1,1,1,1,1,1,1,1,1,1,0,0,0,0,0, \quad g_{26}^A = \epsilon^3 s^3 F_1^A 0,1,1,1,1,1,1,1,1,1,1,0,0,-1,0,0, \]
With the variable $x = t/s$, the differential equations take the following form,

$$\partial_x f(x, \epsilon) = \epsilon \left( \frac{a}{x} + \frac{b}{1 + x} \right) f(x, \epsilon).$$

where $a$ and $b$ are $N \times N$ matrices with constant indices, with $N = 26$ and $N = 41$, respectively for cases A and E.
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The matrices $a$ and $b$ for case A are on the next slide.
Three singularities, at $x = 0$, $x = -1$, and $x = \infty$ corresponding to the limits $s = 0$, $u = 0$, and $t = 0$, respectively.
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A solution near \( D = 4 \) dimensions, so we parametrize, e.g. for family \( A \),

\[
f^A_i(x, \epsilon) = \sum_{j=0}^{6} \epsilon^j f^{A;j}_i(x) + O(\epsilon^7).\]
Three singularities, at $x = 0$, $x = -1$, and $x = \infty$ corresponding to the limits $s = 0$, $u = 0$, and $t = 0$, respectively.

A solution near $D = 4$ dimensions, so we parametrize, e.g. for family $A$,

$$f_i^A(x, \epsilon) = \sum_{j=0}^{6} \epsilon^j f_i^{A;j}(x) + \mathcal{O}(\epsilon^7) .$$

The iterative solution in $\epsilon$ for all functions $f_i$ can be expressed in terms of harmonic polylogarithms of argument $x$ and with indices drawn from $0, -1$, up to boundary constants.
For planar graphs we expect the limit $u \to 0$, i.e. $x \to -1$ to be finite.
The solution should be real for $x > 0$, i.e. when $s$ and $t$ have the same sign.
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The solution should be real for $x > 0$, i.e. when $s$ and $t$ have the same sign.
These conditions fix almost everything: the only additional information needed can easily be obtained from $f_1$:

$$
f_1^A = e^{3\epsilon\gamma_E} \Gamma^4(1 - \epsilon)\Gamma(1 + 3\epsilon) / \Gamma(1 - 4\epsilon)
= 1 - \epsilon^2 \frac{\pi^2}{4} - 29\epsilon^3 \zeta_3 - \epsilon^4 \frac{71}{160} \pi^4 + \epsilon^5 \left( \frac{29}{4} \pi^2 \zeta_3 - \frac{1263}{5} \zeta_5 \right)
+ \epsilon^6 \left( -\frac{11539}{24192} \pi^6 + \frac{841}{2} \zeta_3^2 \right) + O(\epsilon^7).
$$
\begin{align*}
  f_{26}^A(x, \epsilon) &= -\frac{4}{9} + \frac{13\pi^2 \epsilon^2}{36} + \frac{1}{2} \epsilon H\{0\}(x) \\
  &+ \epsilon^3 \left( \frac{9}{4} \pi^2 H\{-1\}(x) - \frac{15}{8} \pi^2 H\{0\}(x) + \frac{9}{2} H\{-1,0,0\}(x) \\
  &\quad \quad \quad \quad \quad - \frac{9}{2} H\{0,0,0\}(x) - \frac{71 \zeta_3}{18} \right) \\
  &+ \epsilon^4 \left( \frac{61\pi^4}{720} + \frac{21}{4} \pi^2 H\{-1,-1\}(x) - \frac{25}{4} \pi^2 H\{-1,0\}(x) \\
  &\quad \quad \quad \quad \quad - \frac{21}{4} \pi^2 H\{0,-1\}(x) + \frac{25}{4} \pi^2 H\{0,0\}(x) \\
  &\quad \quad \quad \quad \quad + \frac{21}{2} H\{-1,-1,0,0\}(x) - 27 H\{-1,0,0,0\}(x) \\
  &\quad \quad \quad \quad \quad - \frac{21}{2} H\{0,-1,0,0\}(x) + 27 H\{0,0,0,0\}(x) + \frac{21}{2} H\{-1\}(x) \zeta_3 \\
  &\quad \quad \quad \quad \quad - 2 H\{0\}(x) \zeta_3 \right) + \ldots
\end{align*}
Two-loop four-point Feynman integrals for Bhabha scattering

(1)

(2a)
Two-loop four-point Feynman integrals for Bhabha scattering

\[ G_{a_1, \ldots, a_4}(s, t, m^2; D) = \int \frac{d^D k}{[-k^2 + m^2]^{a_1}[-(k + p_1)^2]^{a_2}[-(k + p_1 + p_2)^2 + m^2]^{a_3}[-(k - p_3)^2]^{a_4}}, \]
\begin{align*}
G_{a_1, a_2, \ldots, a_9}(s, t, m^2; D) &= \int \int \frac{d^D k_1 d^D k_2}{(-k_1^2 + m^2)^{a_1}[-(k_1 + p_1 + p_2)^2 + m^2]^{a_2}} \\
&\times \frac{[-(k_2 + p_1)^2]^{-a_8}[-(k_1 - p_3)^2]^{-a_9}}{[-k_2^2 + m^2]^{a_3}[-(k_2 + p_1 + p_2)^2 + m^2]^{a_4}[-(k_1 + p_1)^2]^{a_5}[-(k_1 - k_2)^2]^{a_6}[-(k_2 - p_3)^2]^{a_7}}
\end{align*}
\[
G_{a_1,a_2,...,a_9}(s,t,m^2; D) = \int \int \frac{d^D k_1 \, d^D k_2}{(-k_1^2 + m^2)^{a_1} \left[-(k_1 + p_1 + p_2)^2 + m^2\right]^{a_2}}
\times \frac{\left[-(k_2 + p_1)^2\right]^{-a_8} \left[-(k_1 - p_3)^2\right]^{-a_9}}{\left[-k_2^2 + m^2\right]^{a_3} \left[-(k_2 + p_1 + p_2)^2 + m^2\right]^{a_4} \left[-(k_1 + p_1)^2\right]^{a_5} \left[-(k_1 - k_2)^2\right]^{a_6} \left[-(k_2 - p_3)^2\right]^{a_7}}
\]

Results for some of the master integrals for 2a

[VS'02, G. Heinrich & VS'04, M. Czakon, J. Gluza & T. Riemann'04–06]
\[ G_{a_1,a_2,\ldots,a_9}(s,t,m^2;D) = \int \int \frac{d^D k_1 \, d^D k_2}{(-k_1^2 + m^2)^{a_1} \left[-(k_1 + p_1 + p_2)^2 + m^2 \right]^{a_2} \left[-(k_2 + p_1)^2 \right]^{a_8} \left[-(k_1 - p_3)^2 \right]^{a_9} \left[-k_2^2 + m^2 \right]^{a_3} \left[-(k_2 + p_1 + p_2)^2 + m^2 \right]^{a_4} \left[-(k_1 + p_1)^2 \right]^{a_5} \left[-(k_1 - k_2)^2 \right]^{a_6} \left[-(k_2 - p_3)^2 \right]^{a_7}} \]

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\[
\frac{-s}{m^2} = \frac{(1 - x)^2}{x}, \quad \frac{-t}{m^2} = \frac{(1 - y)^2}{y}.
\]
\[ G_{a_1, a_2, \ldots, a_9}(s, t, m^2; D) = \int \int \frac{d^D k_1 \ d^D k_2}{(-k_1^2 + m^2)^{a_1}[-(k_1 + p_1 + p_2)^2 + m^2]^{a_2}} \times \]

\[ \frac{[-(k_2 + p_1)^2]^{-a_8}[-(k_1 - p_3)^2]^{-a_9}}{[-k_2^2 + m^2]^{a_3}[-(k_2 + p_1 + p_2)^2 + m^2]^{a_4}[-(k_1 + p_1)^2]^{a_5}[-(k_1 - k_2)^2]^{a_6}[-(k_2 - p_3)^2]^{a_7}} \]

**Results for some of the master integrals for 2a**

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\]

Due to invariance under inversions of \(x\) and \(y\), it is sufficient to consider \(|x| < 1, |y| < 1\).
Singular points

\[ x = 0 \leftrightarrow s = \infty , \quad x = 1 \leftrightarrow s = 0 \quad x = -1 \leftrightarrow s = 4m^2 \]

A branch cut in the \( s \)-channel starting at \( s = 4m^2 \) and a branch cut in the \( t \)-channel starting at \( t = 0 \).
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No singularity at \( s = 0 \)

The analytic result should be real-valued in the \( s < 0, t < 0 \), i.e. \( 0 < x < 1, 0 < y < 1 \).
\[ f_i = (m^2)^\varepsilon e^{2G_\gamma} g_i \]

with

\[ g_1 = \varepsilon G_{2,0,0,0} , \]
\[ g_2 = \varepsilon t G_{0,2,0,1} , \]
\[ g_3 = \varepsilon \sqrt{(-s)(4m^2 - s)} G_{2,0,1,0} , \]
\[ g_4 = -2\varepsilon^2 (4m^2 - t)(-t) G_{1,1,0,1} , \]
\[ g_5 = -2\varepsilon^2 \sqrt{(-s)(4m^2 - s)t} G_{1,1,1,1} . \]
The normalization is such that

\[ f_i = \sum_{k \geq 0} \epsilon^k f_i^{(k)}. \]
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\[ f_2 = -\epsilon \frac{\Gamma(1 - \epsilon)\Gamma(-\epsilon)\Gamma(1 + \epsilon)}{\Gamma(1 - 2\epsilon)} \left(\frac{y}{(1 - y)^2}\right)^\epsilon e^{\epsilon \gamma_E}. \]
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We obtain

\[ d f = \epsilon d\tilde{A} f \]

with
\[
\tilde{A} = \left[ \begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 \\
\end{array} \right] \log x + \left[ \begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -8 & 0 & -2 & 0 \\
\end{array} \right] \log(1 + x) + \left[ \begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
2 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & -4 & 0 & 0 & 0 \\
\end{array} \right] \log y \\
+ \left[ \begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right] \log(1 + y) + \left[ \begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right] \log(1 - y) + \\
\left[ \begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right] \log(x + y) + \left[ \begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right] \log(1 + xy) \right].
\]
A solution in terms of Chen iterated integrals

\[ f(x, y, \epsilon) = \mathbb{P} e^{\epsilon} \int_{\mathcal{C}} d\tilde{A} \ g(\epsilon), \]

which can be evaluated in terms of multiple polylogarithms. For example,
A solution in terms of Chen iterated integrals

\[ f(x, y, \epsilon) = \mathbb{P}e^{\epsilon} \int_{C} d\tilde{A} \, g(\epsilon), \]

which can be evaluated in terms of multiple polylogarithms. For example,

\[ f_5 = \epsilon \left[ 4H_0(x) \right] + \epsilon^2 \left[ 4G_0(y)H_0(x) - 8G_1(y)H_0(x) \right] \]
\[ + \epsilon^3 \left[ -8G_0(y)H_{-1,0}(x) + 4G_0(y)H_{0,0}(x) - 8H_0(x)G_{1,0}(y) + 16H_0(x)G_{1,1}(y) \right] \]
\[ + 4H_0(x)G_{-\frac{1}{x},0}(y) - 8H_0(x)G_{-\frac{1}{x},1}(y) + 4H_0(x)G_{-x,0}(y) - 8H_0(x)G_{-x,1}(y) \]
\[ + 8H_{-1,0}(x)G_{-\frac{1}{x},0}(y) + 8H_{-1,0}(x)G_{-x}(y) - 4H_{0,0}(x)G_{-\frac{1}{x},0}(y) - 4H_{0,0}(x)G_{-x}(y) \]
\[ + 4G_{-\frac{1}{x},0,0}(y) - 8G_{-\frac{1}{x},0,1}(y) - 4G_{-x,0,0}(y) + 8G_{-x,0,1}(y) + 8H_{-2,0}(x) \]
\[ - 16H_{-1,-1,0}(x) + 8H_{-1,0,0}(x) - 4H_{0,0,0}(x) + \frac{10}{3} \pi^2 G_{-\frac{1}{x},0}(y) - 2\pi^2 G_{-x}(y) \]
\[ - \frac{2}{3} \pi^2 G_0(y) - \frac{4}{3} \pi^2 H_{-1}(x) - \frac{7}{3} \pi^2 H_0(x) + 8\zeta_3 \right] + \mathcal{O}(\epsilon^4). \]
\[ G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t), \]

with

\[ G(a_1; z) = \int_0^z \frac{dt}{t - a_1}, \quad a_1 \neq 0. \]

For \( a_1 = 0 \), we have \( G(\vec{0}_n; x) = \frac{1}{n!} \log^n(x). \)
(17),(18)†  (19)  (20),(21)†  (22),(23)*
\[ df = \epsilon \, d\tilde{A} \, f \]

with
\[ d f = \epsilon \, d\tilde{A} \, f \]

with

\[ \tilde{A} = B_1 \log(x) + B_2 \log(1 + x) + B_3 \log(1 - x) + B_4 \log(y) + B_5 \log(1 + y) \]
\[ + B_6 \log(1 - y) + B_7 \log(x + y) + B_8 \log(1 + xy) \]
\[ + B_9 \log(x + y - 4xy + x^2y + xy^2) + B_{10} \log\left(\frac{1 + Q}{1 - Q}\right) \]
\[ + B_{11} \log\left(\frac{(1 + x) + (1 - x)Q}{(1 + x) - (1 - x)Q}\right) + B_{12} \log\left(\frac{(1 + y) + (1 - y)Q}{(1 + y) - (1 - y)Q}\right) \]
\[ \mathbf{d} f = \epsilon \mathbf{d} \tilde{A} f \]

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\[ Q = \sqrt{\frac{(x + y)(1 + xy)}{x + y - 4xy + x^2y + xy^2}}, \]
At order $\epsilon, \epsilon^2$ and $\epsilon^3$, the arguments of the logarithms in $\tilde{A}$ are the same as at one loop.

At order $\epsilon^4$ all basis functions except $f_{11}$ have arguments as at one loop.
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At order $\epsilon^4$ all basis functions except $f_{11}$ have arguments as at one loop.

For example,
\[ f_{23} = \epsilon^2 \left[ -12 H_{0,0}(x) \right] + \epsilon^3 \left[ -16 G_0(y) H_{0,0}(x) + 32 G_1(y) H_{0,0}(x) + 8 H_{2,0}(x) \right. \\
+ 16 H_{-1,0,0}(x) + 4 H_{0,0,0}(x) + \frac{4}{3} \pi^2 H_0(x) + 4 \zeta_3 \right] + \epsilon^4 \left[ 32 G_0(y) H_{-2,0}(x) \right. \\
- 32 H_{-2,0}(x) G_{\frac{1}{x},0}(y) - 32 H_{-2,0}(x) G_{-x}(y) + 64 G_{1,0}(y) H_{0,0}(x) - 128 G_{1,1}(y) H_{0,0}(x) \\
- 32 H_{0,0}(x) G_{\frac{1}{x},0}(y) + 64 H_{0,0}(x) G_{\frac{1}{x},0,1}(y) - 32 H_{0,0}(x) G_{-x,0}(y) \\
+ 64 H_{0,0}(x) G_{-x,1}(y) - 16 H_{0}(x) G_{\frac{1}{x},0,0}(y) + 32 H_{0}(x) G_{\frac{1}{x},0,1}(y) \\
+ 16 H_{0}(x) G_{-x,0,0}(y) - 32 H_{0}(x) G_{-x,0,1}(y) + 64 G_{0}(y) H_{-1,0,0}(x) \\
- 64 H_{-1,0,0}(x) G_{\frac{1}{x},0}(y) - 64 H_{-1,0,0}(x) G_{-x}(y) \\
- 48 G_{0}(y) H_{0,0,0}(x) + 48 H_{0,0,0}(x) G_{\frac{1}{x},0}(y) + 48 H_{0,0,0}(x) G_{-x}(y) - 120 H_{-3,0}(x) \\
+ \frac{52}{3} \pi^2 H_{0,0}(x) + 48 H_{3,0}(x) + 128 H_{-2,-1,0}(x) - 120 H_{-2,0,0}(x) - 48 H_{-2,1,0}(x) \\
+ 64 H_{-1,-2,0}(x) - 32 H_{-1,2,0}(x) - 48 H_{2,-1,0}(x) + 32 H_{2,0,0}(x) + 16 H_{2,1,0}(x) \\
+ 64 H_{-1,-1,0,0}(x) - 80 H_{-1,0,0,0}(x) + 76 H_{0,0,0,0}(x) + \frac{8}{3} \pi^2 G_0(y) H_0(x) \\
- \frac{40}{3} \pi^2 H_0(x) G_{\frac{1}{x},0}(y) + 8 \pi^2 H_0(x) G_{-x}(y) - 16 \zeta_3 H_{-1}(x) - 28 \zeta_3 H_0(x) \\
+ \frac{8}{3} \pi^2 H_{-2}(x) - \frac{4}{3} \pi^2 H_2(x) - \frac{4 \pi^4}{15} \right] + \mathcal{O}(\epsilon^5) \]
Evaluating single-scale diagrams by DE
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A three-loop form-factor integral called $A_{92}$.

[J.M. Henn, A. Smirnov & V.S.’13]
Evaluating single-scale diagrams by DE

A three-loop form-factor integral called $A_{92}$.

$$p_2^2 = 0 \rightarrow p_2^2 \neq 0, \text{ with } x = p_2^2/q^2.$$
$K_4$
A straightforward strategy to evaluate Feynman integrals: integrate consecutively over Feynman parameters

[F. Brown’09]
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Results obtained within this scenario

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Computer codes to perform such parametric integration

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An attempt to evaluate $K_4$ with all the external momenta on the light cone

[C. Bogner & M. Lüders’13]
\[ F^C_{a_1, \ldots, a_{15}}(s, t; D) = \frac{1}{(i\pi^{D/2})^3} \int \int \int \frac{d^D k_1 \, d^D k_2 \, d^D k_3}{(-k_1^2)^{a_1} \, (-p_1 + p_2 + k_1)^2)^{a_2} \, (-k_1 + k_3)^2)^{a_3}} \times \frac{[-(k_1 + k_2)^2)^{a_11} \, (-p_1 + k_3)^2)^{a_12} \, (-p_1 + k_2)^2)^{a_13}}{[-(p_1 + p_2 + k_1 + k_2)^2)^{a_4} \, (-k_1 + k_2 + k_3)^2)^{a_5} \, (-p_1 + p_2 + k_1 + k_2 + k_3)^2)^{a_6}} \times \frac{[-(p_3 + k_1)^2)^{a_14} \, (-p_3 + k_3)^2)^{a_15}}{(-k_3^2)^{a_7} \, (-k_2^2)^{a_8} \, (-p_1 + k_1)^2)^{a_9} \, (-k_1 + k_2 + k_3 - p_3)^2)^{a_{10}}} . \]
\[ F_{a_1, \ldots, a_{15}}^C(s, t; D) = \frac{1}{(i\pi^{D/2})^3} \int \int \int \frac{d^D k_1 \, d^D k_2 \, d^D k_3}{(-k_1^2)^{a_1} \, [-(p_1 + p_2 + k_1)^2]^{a_2} \, [-(k_1 + k_3)^2]^{a_3}} \times \frac{[(-(k_3 + k_2)^2]^{a_{11}} \, [-(p_1 + k_3)^2]^{a_{12}} \, [-(p_1 + k_2)^2]^{a_{13}}}{[-(p_1 + p_2 + k_1 + k_2)^2]^{a_4} \, [-(k_1 + k_2 + k_3)^2]^{a_5} \, [-(p_1 + p_2 + k_1 + k_2 + k_3)^2]^{a_6}} \times \frac{[-(p_3 + k_1)^2]^{a_{14}} \, [-(p_3 + k_3)^2]^{a_{15}}}{(-k_3^2)^{a_7} \, (-k_2^2)^{a_8} \, [-(p_1 + k_1)^2]^{a_9} \, [-(k_1 + k_2 + k_3 - p_3)^2]^{a_{10}}}. \]

\[ K_{a_1, a_2, \ldots, a_6} = F_{0, 0, a_1, a_2, 0, 0, a_3, a_4, a_5, a_6, 0, \ldots, 0}^C. \]

\[ \hat{K}_{a_1, a_2, \ldots, a_6, a'} = F_{0, a', a_1, a_2, 0, 0, a_3, a_4, a_5, a_6, 0, \ldots, 0}^C, \]

where \( a' \leq 0 \).
We choose a UT basis

\[
f = e^{3\varepsilon \gamma_E} (-s)^{-3\varepsilon} g = (f_1, f_2, \ldots, f_{10})
\]

\[
g_1 = \varepsilon^3 t K_{0,0,1,2,2,2}, \quad g_2 = \varepsilon^3 (s + t) K_{1,2,0,0,2,2}, \\
g_3 = \varepsilon^3 s K_{1,2,2,2,0,0}, \quad g_4 = 2\varepsilon^4 (s + t) \hat{K}_{1,2,1,1,2,1,1} - 1 + 2\varepsilon^5 s K_{2,1,1,1,1,1}, \\
g_5 = 4\varepsilon^5 t K_{2,1,1,1,1,1}, \quad g_6 = 4\varepsilon^5 (s + t) K_{1,1,2,1,1,1}, \\
g_7 = 4\varepsilon^5 s K_{1,1,1,1,2,1}, \quad g_8 = -2\varepsilon^4 s (s + t) K_{2,2,1,1,1,1}, \\
g_9 = -2\varepsilon^4 st K_{1,1,2,2,1,1}, \quad g_{10} = -2\varepsilon^4 (s + t) t K_{1,1,1,1,2,2},
\]
We choose a UT basis

\[ f = e^{3\epsilon\gamma E} (-s)^{-3\epsilon} g = (f_1, f_2, \ldots, f_{10}) \]

\[
g_1 = \epsilon^3 t K_{0,0,1,2,2,2}, \quad g_2 = \epsilon^3 (s + t) K_{1,2,0,0,2,2},
\]
\[
g_3 = \epsilon^3 s K_{1,2,2,2,0,0}, \quad g_4 = 2\epsilon^4 (s + t) \hat{K}_{1,2,1,1,2,1,-1} + 2\epsilon^5 s K_{2,1,1,1,1,1},
\]
\[
g_5 = 4\epsilon^5 t K_{2,1,1,1,1,1}, \quad g_6 = 4\epsilon^5 (s + t) K_{1,1,2,1,1,1},
\]
\[
g_7 = 4\epsilon^5 s K_{1,1,1,1,2,1}, \quad g_8 = -2\epsilon^4 s (s + t) K_{2,2,1,1,1,1},
\]
\[
g_9 = -2\epsilon^4 st K_{1,1,2,2,1,1}, \quad g_{10} = -2\epsilon^4 (s + t) t K_{1,1,1,1,2,2},
\]

DE

\[
\partial_x f(x, \epsilon) = \epsilon \left[ \frac{A}{x} + \frac{B}{1 + x} \right] f(x, \epsilon).
\]
$$A = \begin{pmatrix}
-3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{2}{3} & \frac{2}{3} & -\frac{1}{6} & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{7}{6} & \frac{1}{12} & -\frac{1}{12} & \frac{1}{3} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & 4 & 5 & -3 & -3 & -\frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{3} & \frac{5}{3} & \frac{1}{3} & 4 & \frac{7}{3} & -\frac{7}{3} & -\frac{11}{3} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\
-\frac{4}{3} & \frac{10}{3} & -\frac{10}{3} & 0 & \frac{20}{3} & \frac{10}{3} & -\frac{10}{3} & \frac{5}{3} & -\frac{2}{3} & \frac{2}{3} \\
-\frac{14}{3} & \frac{8}{3} & \frac{4}{3} & 8 & \frac{22}{3} & -\frac{16}{3} & -\frac{20}{3} & -\frac{2}{3} & -\frac{7}{3} & \frac{4}{3} \\
\frac{10}{3} & \frac{8}{3} & \frac{4}{3} & 8 & \frac{22}{3} & -\frac{16}{3} & -\frac{20}{3} & -\frac{2}{3} & \frac{2}{3} & -\frac{5}{3}
\end{pmatrix},$$

$$B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & -5 & -\frac{7}{3} & \frac{5}{6} & \frac{1}{6} & \frac{1}{12} & -\frac{1}{12} \\
-\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 4 & \frac{5}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{3} & -\frac{5}{3} & -\frac{1}{3} & -4 & -\frac{7}{3} & \frac{7}{3} & -\frac{1}{3} & \frac{1}{6} & -\frac{1}{6} & -\frac{1}{3} \\
0 & -2 & 0 & 8 & 4 & -2 & 0 & -3 & 0 & 0 \\
\frac{10}{3} & -\frac{4}{3} & \frac{10}{3} & 0 & \frac{10}{3} & \frac{20}{3} & -\frac{2}{3} & \frac{5}{3} & -\frac{2}{3} \\
0 & -6 & 0 & -8 & -4 & 2 & 0 & 0 & 0 & -3
\end{pmatrix}.$$
The first three integrals can be expressed in terms of gamma functions.
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Using the limit $x \to 0$ to fix boundary conditions.

An interplay with expansion by regions which gives contributions with $x^{0\epsilon}$ (h-h-h) and $x^{-3\epsilon}$ (c-c-c).
The first three integrals can be expressed in terms of gamma functions.

Using the limit $x \to 0$ to fix boundary conditions.

An interplay with expansion by regions which gives contributions with $x^{0\epsilon} \ (h-h-h)$ and $x^{-3\epsilon} \ (c-c-c)$.

Terms with $x^{k\epsilon}$ at $k > 0$ are absent!

It was possible to evaluate the LO (c-c-c) terms analytically. For example, for $K_{2,2,1,1,1,1,1}$:

$$x^{-3\epsilon} \left[ -\frac{421}{5} \zeta_5 \log(x) + \frac{29}{12} \pi^2 \zeta_3 \log(x) - \frac{421 i \pi \zeta_5}{10} + \frac{5597 \zeta(3)^2}{36} ight. \\
+ \frac{29}{24} i \pi^3 \zeta_3 + \frac{31601 \pi^6}{2177280} + O(x) \right].$$
An example of result

\[ K^{(0)}(x, \epsilon) = e^{3\epsilon \gamma E} (-s)^{-3\epsilon} (1 - 4\epsilon)(1 - 5\epsilon) \epsilon^4 K_{1,1,1,1,1,1}(x, \epsilon), \]
An example of result

\[ K^{(0)}(x, \epsilon) = e^{3\epsilon\gamma_E} (-s)^{-3\epsilon} (1 - 4\epsilon)(1 - 5\epsilon) \epsilon^4 K_{1,1,1,1,1,1}(x, \epsilon), \]

\[ K^{(0)}(x, \epsilon) = 2\zeta_3 \epsilon^3 \]
\[ + \epsilon^4 \left[ 3i\pi \zeta_3 + \frac{3\pi^4}{20} + 2i\pi H_{-3}(x) + \frac{1}{2}\pi^2 H_{-2}(x) - \frac{1}{2}i\pi^3 H_{-1}(x) - 3H_{-1}(x)\zeta_3 \right. \]
\[ - 2H_{-3,-1}(x) + H_{-2,-2}(x) - i\pi H_{-2,0}(x) + H_{-1,-3}(x) - \pi^2 H_{-1,-1}(x) \]
\[ + \frac{1}{2}\pi^2 H_{-1,0}(x) + H_{-2,-1,0}(x) + H_{-1,-2,0}(x) - i\pi H_{-1,0,0}(x) - 2H_{-1,-1,0,0}(x) \]
\[ + \mathcal{O}(\epsilon^5). \]
Massless four-point integrals with two off-shell legs
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NNLO QCD corrections to the production of two off-shell vector bosons in hadron collisions.
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NNLO QCD corrections to the production of two off-shell vector bosons in hadron collisions. Planar diagrams
Massless four-point integrals with two off-shell legs

NNLO QCD corrections to the production of two off-shell vector bosons in hadron collisions.
Planar diagrams

\[ p_1 \rightarrow 1 \rightarrow 3 \rightarrow p_3 \]
\[ 5 \rightarrow 6 \rightarrow 7 \]
\[ p_2 \rightarrow 2 \rightarrow 4 \rightarrow p_4 \]
Massless four-point integrals with two off-shell legs

NNLO QCD corrections to the production of two off-shell vector bosons in hadron collisions.
Planar diagrams

$P_{12}: p_1 = -q_3, \ p_2 = -q_4, \ p_3 = q_1, \ p_4 = q_2$;

$P_{13}: p_1 = -q_3, \ p_2 = q_1, \ p_3 = -q_4, \ p_4 = q_2$;

$P_{23}: p_1 = q_2, \ p_2 = -q_4, \ p_3 = -q_3, \ p_4 = q_1$.

where $q_1^2 = 0, q_2^2 = 0$ and $q_3^2 = M_3^2, q_4^2 = M_4^2$.
Nonplanar diagrams
Nonplanar diagrams
Nonplanar diagrams

$N_{12}: \ p_1 = -q_4, \ p_2 = -q_3, \ p_3 = q_2, \ p_4 = q_1$;

$N_{13}: \ p_1 = -q_4, \ p_2 = q_2, \ p_3 = -q_3, \ p_4 = q_1$;

$N_{34}: \ p_1 = q_1, \ p_2 = q_2, \ p_3 = -q_3, \ p_4 = -q_4$. 
Evaluation of the planar diagrams in the equal mass case

[T. Gehrmann, L. Tancredi & E. Weihs’13]
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For general $M_3^2, M_4^2$  

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Non-planar diagrams in the equal mass case
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For general $M^2_3, M^2_4$ [F. Caola, J.M. Henn, K. Melnikov & V.A. Smirnov’14]
Get rid of a square root:

\[ \frac{S}{M_3^2} = (1 + x)(1 + xy), \quad \frac{T}{M_3^2} = -xz, \quad \frac{M_4^2}{M_3^2} = x^2 y. \]
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Then

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In terms of \(x, y, z\), the physical region is

\[ x > 0, \quad y > 0, \quad y < z < 1. \]
\[ G_{a_1, \ldots, a_9} = \int \int \frac{d^D k'_1 \ d^D k'_2}{[-k_1^2]^{a_1} [-(k_1 + p_1 + p_2)^2]^{a_2} [-k_2^2]^{a_3} [-(k_2 + p_1 + p_2)^2]^{a_4} \times \left[ -(k_2 + p_1)^2 \right]^{-a_8} \left[ -(k_1 - p_3)^2 \right]^{-a_9}} \left[ -(k_1 + p_1)^2 \right]^{a_5} \left[ -(k_1 - k_2)^2 \right]^{a_6} \left[ -(k_2 - p_3)^2 \right]^{a_7} \]
\[ G_{a_1,\ldots,a_9} = \int \int \frac{d^D k_1 \ d^D k_2}{[-k_1^2]^{a_1}[-(k_1 + p_1 + p_2)^2]^{a_2}[-k_2^2]^{a_3}[-(k_2 + p_1 + p_2)^2]^{a_4}} \times \frac{[-(k_2 + p_1)^2]^{-a_8}[-(k_1 - p_3)^2]^{-a_9}}{[-(k_1 + p_1)^2]^{a_5}[-(k_1 - k_2)^2]^{a_6}[-(k_2 - p_3)^2]^{a_7}} \]
\[ \partial_\xi f = \epsilon A_\xi f, \]

where \( \xi = x, y \) or \( z \).
DE

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where \( \xi = x, y \) or \( z \).

In differential form

\[ df(x, y, z; \epsilon) = \epsilon (d \tilde{A}(x, y, z)) f(x, y, z; \epsilon), \]

where the differential \( d \) acts on \( x, y \) and \( z \).
\[ \partial_{\xi} f = \epsilon A_{\xi} f, \]

where \( \xi = x, y \) or \( z \).

In differential form

\[ d f(x, y, z; \epsilon) = \epsilon (d \tilde{A}(x, y, z)) f(x, y, z; \epsilon), \]

where the differential \( d \) acts on \( x, y \) and \( z \).

\[ \tilde{A} = \sum_{i=1}^{15} \tilde{A}_{\alpha_i} \log(\alpha_i), \]

where the \( \tilde{A}_{\alpha_i} \) are constant matrices.
In the *planar* case, 
the arguments of the logarithms $\alpha_i$ (*letters*) are

$$\alpha = \{ x, y, z, 1 + x, 1 - y, 1 - z, 1 + xy, z - y, \\
1 + y(1 + x) - z, xy + z, 1 + x(1 + y - z), 1 + xz, 1 + y - z, \\
z + x(z - y) + xyz, z - y + yz + xyz \}$$

with only a linear dependence on $x, y, z$. 
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Solving DE iteratively order-by-order in $\epsilon$

$$f = \sum_{n=0}^{4} f^{(n)} \epsilon^n + \mathcal{O}(\epsilon^5).$$
In the planar case, the arguments of the logarithms $\alpha_i$ (letters) are

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Solving DE iteratively order-by-order in $\epsilon$

$$f = \sum_{n=0}^{4} f(n) \epsilon^n + O(\epsilon^5).$$

Integrate first in $x$, then in $y$, then in $z$. 
We have decided to obtain results directly in the physical region $x > 0, y > 0, y < z < 1$ because it is difficult to perform an analytic continuation from a Euclidean region due to complicated dependence of $x, y, z$ on the kinematic invariants.
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\( x \to 0, z \to 1, y \to 1 \)
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For \( P_{12} \) and \( P_{13} \) one can immediately set \( y = z = 1 \) and a typical behaviour in this limit is \( f \sim f_a x^{-n_\alpha \epsilon} \)
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For \( P_{23} \) (and for all non-planar families) the limit \( y, z \to 1 \) is singular, with a typical behaviour

\[
f \sim f_a x^{-n_1 \epsilon} + f_b x^{-n_2 \epsilon} [(z - y)(1 - z)]^{-n_3 \epsilon}
\]
To evaluate the LO asymptotics in the limit $x \to 0$, $z \to 1$, $y \to 1$ we applied expansion by regions

[M. Beneke & V.S.'98]

implemented in the open computer code $\text{asy.m}$

[A. Pak & A. Smirnov’11, B. Jantzen, A. Smirnov & VS’12]

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Evaluating these integrals by Mellin–Barnes representation.

Obtaining the boundary conditions also from the consistency of DE.
Because of a linear dependence of the letters on $x, y, z$, results can be expressed in terms of multiple (Goncharov) polylogarithms

$$G(a_1, \ldots a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t),$$

with

$$G(a_1; z) = \int_0^z \frac{dt}{t - a_1}, \quad a_1 \neq 0.$$

For $a_1 = 0$, we have $G(\vec{0}_n; x) = \frac{1}{n!} \log^n(x)$. 
\[
\begin{align*}
g_{28}^{P23} &= \epsilon^2 \left( 2\epsilon p_2^2 (p_3^2 - s) G_{1,0,0,1,2,1,0,0} - 4\epsilon p_2^2 (p_3^2 - s) G_{1,1,0,0,1,1,2,0,0} \\
&\quad + 4\epsilon^2 s (-p_3^2 + s) G_{1,1,1,1,1,1,1,-1,0} \right) = \\
&\quad -(\epsilon p^2 (11\pi^2 - (18i) \pi G[0, z] + G[0, y] ((6i) \pi + 6G[0, z]) - (6i) \pi G[1, z] - \\
&\quad 6G[0, y] G[-((1 + y - z)/y), x] + 6G[1, z] (G[-((1 + y - z)/y), x] + \\
&\quad G[0, x] ((-12i) \pi + 6G[0, y] - 6G[1, z] - \\
&\quad 6G[-1 + z, y]) + ((6i) \pi + 6G[1, z]) G[-1 + z, y] + \\
&\quad G[-((1 + y - z)/y), x] G[(6i) \pi + 6G[-1 + z, y]) + \\
&\quad (6i) \pi G[-z^(-1), x] + 6G[0, z] (G[-z^(-1), x] - \\
&\quad (12i) \pi G[z, y] - 6G[0, z] G[z, y] - 6G[1, z] G[z, y] - \\
&\quad 12G[0, 0, z] - 6G[0, 1, z] - 6G[1, 0, z] - \\
&\quad 6G[-((1 + y - z)/y), 0, x] - 6G[-1 + z, 0, y] + \\
&\quad 6G[-1 + z, -1 + z, y] + 6G[-z^(-1), 0, x] + 6G[z, 0, y] - \\
&\quad 6G[z, -1 + z, y])) - \epsilon p^3 ((16i/3) \pi^3 G[-z^(-1), x]) / 3 + \\
&\quad (\pi^2 G[0, x]) / 3 - (19\pi^2 G[0, z]) / 3 - (31\pi^2 G[1, z]) / 3 + \\
&\quad (\pi^2 G[-z^(-1), x]) / 3 - (\pi^2 G[z, y]) / 3 - (24i) \pi G[-1, 0, x] + \\
&\quad (12i) \pi G[-1, -1 + y - z) ^(-1), x] + \\
&\quad (12i) \pi G[-1, -(1 + y - z)/y), x] + (12i) \pi G[-1, -z^(-1), x] + \\
&\quad (12i) \pi G[-1, -(z/y), x] + (80i) \pi G[0, 0, x] - \\
&\quad (6i) \pi G[0, 0, y] + G[-(z/y), x] (8\pi^2 - 16G[0, 0, z]) + \\
&\quad (34i) \pi G[0, 0, z] + (24i) \pi G[0, 1, z] - \\
&\quad (24i) \pi G[0, -(1 + y - z)^(-1), x] + ...
\end{align*}
\]
Checks:

- Comparison with the equal mass case, $M_3^2 = M_4^2 = M^2$

[T. Gehrmann, L. Tancredi & E. Weihs'13]
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- Using FIESTA3
  [A. Smirnov'13]

- Using analytic continuation over contour in the complex plane starting from a point in an unphysical region where the boundary conditions are simple.
For the *nonplanar* families $N_{12}$ and $N_{13}$ we choose the same parametrization as in the planar case

\[ S = M^2(1+x)(1+xy), \quad T = -M^2xz, \quad M_3^2 = M^2, \quad M_4^2 = M^2x^2y \]
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For $N_{34}$ we choose

$$S = M^2(1 + x)^2, \quad T = -M^2x((1 + y)(1 + xy) - 2zy(1 + x)),$$
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The physical region is

$$x < 1/y, \quad 0 < y < 1, \quad 0 < z < 1.$$
For $N_{12}$ and $N_{13}$, we have the following letters

$$\{x, 1 + x, 1 - y, y, 1 + xy, 1 + x(1 + y - z), 1 - z, y - z, 1 + y - z, 1 + y + xy - z, z, -y + z, xy + z, 1 + x + xy - xz, 1 + xz, 1 + y + 2xy - z + x^2yz, z - y(1 - z - xz)\}$$
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\[
\{x, 1 + x, 1 - y, y, 1 + xy, 1 + x(1 + y - z), 1 - z, y - z, 1 + y - z, \\
1 + y + xy - z, z, -y + z, xy + z, 1 + x + xy - xz, 1 + xz, \\
1 + y + 2xy - z + x^2yz, z - y(1 - z - xz)\}
\]

For $N_{34}$, we have

\[
\{x, 1 + x, 1 - y, y, 1 + y, 1 - xy, 1 + xy, 1 - y(1 - 2z), 1 + y - 2yz, \\
1 - xy^2 - y(1 - x - 2z + 2xz), 1 - xy(1 - 2z), 1 + x(y - 2yz), \\
1 + xy^2 - (1 + x)y(1 - 2z), 1 - z, z, 1 + y - 2yz, \\
(1 + y)(1 + xy) - 2zy(1 + x), 1 - y + 2yz, \\
1 - xy^2 + (1 - x)y(1 - 2z)\}
\]
In contrast to planar master integrals, there is a quadratic dependence of the letters (for $x$ for $N_{13}$ and for $y$ for $N_{34}$).
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In the physical region, all the letters are sign-definite. All iterated integrals needed for calculating the vector of the master integrals can be written in a manifestly real form, so that imaginary parts appear only through explicit factors of $i$ coming from the boundary conditions.
Conclusion

With this strategy of the method of DE based on UT, it was possible to evaluate families of quite complicated Feynman integrals. (Fließbandarbeit. Mass-line production.)
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- It is also possible to evaluate single scale diagrams.
- Further results will be obtained in the nearest future.
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With this strategy of the method of DE based on UT, it was possible to evaluate families of quite complicated Feynman integrals. (Fließbandarbeit. Mass-line production.)

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It is also possible to evaluate single scale diagrams.

Further results will be obtained in the nearest future.

The method is under construction.