Symmetries Related to AdS/CFT Integrability

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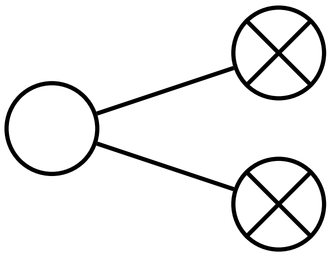
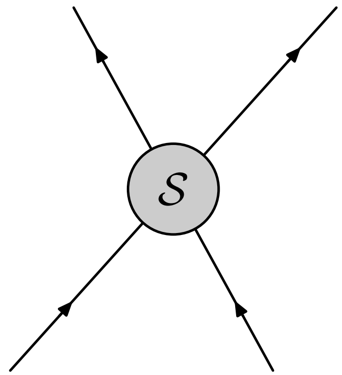




Workshop on Integrability in
Gauge and String Theory

Paris, June 20, 2007

Collaborations with P. Koroteev, F. Spill, B. Zwiebel.
References: `nlin.SI/0610017`, `arxiv:0704.0400`, work in progress.

Introduction

Planar $\mathcal{N} = 4$ SYM & strings on $AdS_5 \times S_5$ are presumably integrable.
Integrable models are undoubtedly very nice physical models.

Moreover: Integrability is a hidden **symmetry**.
Understand symmetry algebra to understand the model.

This talk: Loosely connected aspects of symmetries, mostly $\mathfrak{psl}(2|2)$.

Outline:

- Lie Symmetries
- Affine Algebras and Deformations
- Yangian
- Quantum Deformations
- Degeneracies in the $\mathfrak{psu}(1,1|2)$ Sector.
- Conclusions

Symmetry

AdS/CFT Particle Model(s)

String Theory: Light cone gauge using AdS_5-time and S^5-great circle.

- Vacuum: Point-particle moving along time and great circle. [Berenstein Maldacena Nastase]
- Excitations: 4 coordinates on AdS_5 and 4 coordinates on S^5.
- Fermions: 32 coordinates, $1/2$ momenta, $1/2$ gauged away, 8 remain.

Gauge Theory: Spin chain states with few "excitations". [Staudacher hep-th/0412188]

- Vacuum: Half-BPS state $|0\rangle = |\ldots \mathcal{Z}\mathcal{Z}\mathcal{Z}\ldots\rangle$ (ferromagnetic vacuum).
- One-excitation states with excitation $\mathcal{A}$ of momentum p

$$|\mathcal{A}, p\rangle = \sum_a e^{ipa} |\ldots \mathcal{Z} \ldots \overset{\overset{a}{\downarrow}}{\mathcal{A}} \ldots \mathcal{Z} \ldots\rangle, \qquad \delta\mathcal{H}\,|\mathcal{A}, p\rangle = \delta E_{\mathcal{A}}(p)|\mathcal{A}, p\rangle.$$

  $(4+4|\,4+4)$ flavours of excitations $\mathcal{A} \in \{\phi_i, \mathcal{D}_\mu \mathcal{Z} |\, \psi_a, \dot{\psi}_{\dot{a}}\}$. [Berenstein Maldacena Nastase]
- Other spin orientations $\mathbb{V}_{\mathrm{F}}$ are multiple coincident excitations.

QM particle model of 8 bosonic and 8 fermionic flavours on the circle.

Residual Symmetry $\mathfrak{psu}(2|2)$

Excitations transform as $(\mathbf{2}|\mathbf{2}) \times (\mathbf{2}|\mathbf{2})$ of $\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2)$.

Consider just $(2|2)$ flavours and one copy of $\mathfrak{psu}(2|2)$. Generators:

- $\mathfrak{R}^a{}_b$: $\mathfrak{su}(2)$ subalgebra of S^5/internal symmetry.
- $\mathfrak{L}^\alpha{}_\beta$: $\mathfrak{su}(2)$ subalgebra of AdS_5/conformal symmetry.
- $\mathfrak{Q}^\alpha{}_b$: 4 (Poincaré) supercharges.
- $\mathfrak{S}^a{}_\beta$: 4 (conformal) supercharges.

$\mathfrak{psu}(2|2)$ has three-dimensional (exceptional!) central extension.

Need this central extension $\mathfrak{h} := \mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ for consistency: [NB hep-th/0511082]

- $\mathfrak{C}$: Hamiltonian/dilatation generator (up to integer shift).
- $\mathfrak{P}$: world sheet shift in σ/(classical) gauge variation.
- $\mathfrak{K}$: world sheet shift in σ/(quantum) gauge variation.

Lie Algebra $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$

Lie superalgebra defined by Lie brackets

- $\mathfrak{R}^a{}_b, \mathfrak{L}^\alpha{}_\beta$: canonical brackets of $\mathfrak{su}(2) \times \mathfrak{su}(2)$ generators.
- $\mathfrak{C}, \mathfrak{P}, \mathfrak{K}$: central charges.
- $\mathfrak{Q}^\alpha{}_b, \mathfrak{S}^a{}_\beta$: supercharges

$$\{\mathfrak{Q}^\alpha{}_b, \mathfrak{S}^c{}_\delta\} = \delta^c_b \mathfrak{L}^\alpha{}_\delta + \delta^\alpha_\delta \mathfrak{R}^c{}_b + \delta^c_b \delta^\alpha_\delta \mathfrak{C},$$

$$\{\mathfrak{Q}^\alpha{}_b, \mathfrak{Q}^\gamma{}_d\} = \varepsilon^{\alpha\gamma} \varepsilon_{bd} \mathfrak{P},$$

$$\{\mathfrak{S}^a{}_\beta, \mathfrak{S}^c{}_\delta\} = \varepsilon^{ac} \varepsilon_{\beta\delta} \mathfrak{K}.$$

Fundamental Representation

Have $(2|2)$ flavours of particles $\{|\phi^a\rangle, |\psi^\alpha\rangle\}$. Represent algebra! [NB hep-th/0511082]

Most general action compatible with $\mathfrak{su}(2) \times \mathfrak{su}(2)$

$$\mathfrak{Q}^\alpha{}_b|\phi^c\rangle = a\,\delta^c_b|\psi^\alpha\rangle, \qquad \mathfrak{S}^a{}_\beta|\phi^c\rangle = c\,\varepsilon^{ac}\varepsilon_{\beta\delta}|\mathcal{Z}^-\psi^\delta\rangle,$$
$$\mathfrak{Q}^\alpha{}_b|\psi^\gamma\rangle = b\,\varepsilon^{\alpha\gamma}\varepsilon_{bd}|\mathcal{Z}^+\phi^d\rangle, \qquad \mathfrak{S}^a{}_\beta|\psi^\gamma\rangle = d\,\delta^\gamma_\beta|\phi^a\rangle.$$

Markers $\mathcal{Z}^\pm$ represent (dynamic) insertion/deletion of vacuum field $\mathcal{Z}$

- derived from action of supercharges in gauge theory,
- $\mathfrak{P}, \mathfrak{K}$ are gauge transformations iff $P, K \sim (1 - e^{\pm ip})$.

Imposing consistency of superalgebra

- fixes central charges $C = \frac{1}{2}(ad + bc)$, $P = ab$, $K = cd$,
- yields constraint $ad - bc = 1$ or $C^2 - PK = \frac{1}{4}$,
- provides dispersion relation $C^2 = \frac{1}{4} + 4g^2 \sin^2(\frac{1}{2}p)$.

Coproduct

Coproduct Δ defines how some generator $\mathfrak{J}^A$ acts on multi-particle states.
Coproduct $\Delta : \mathrm{U}(\mathfrak{h}) \to \mathrm{U}(\mathfrak{h}) \otimes \mathrm{U}(\mathfrak{h})$ adds one site. Chain: $\Delta^{L-1}(\mathfrak{J})$.
Trivial coproduct (tensor product action): $\Delta(\mathfrak{J}^A) = \mathfrak{J}^A \otimes 1 + 1 \otimes \mathfrak{J}^A$.
Action of markers $\mathcal{Z}^\pm$ equivalent to non-trivial coproduct [Gomez Hernández][Plefka Spill Torrielli]

$$\Delta(\mathfrak{J}^A) = \mathfrak{J}^A \otimes 1 + \mathfrak{U}^{[A]} \otimes \mathfrak{J}^A.$$

Gradings: $[\mathfrak{P}] = +2$, $[\mathfrak{Q}] = +1$, $[\mathfrak{S}] = -1$, $[\mathfrak{K}] = -2$.
Abelian braiding generator $\mathfrak{U}$ measures momentum $e^{ip/2}$.
Coproduct of braiding element: $\Delta(\mathfrak{U}) = \mathfrak{U} \otimes \mathfrak{U}$.

Non-trivial coproduct due to

- length of the spin chain changing or
- non-locality in $x_- = \int d\sigma\, x'_-$ for string light cone gauge. [Arutyunov, Frolov Plefka, Zamaklar]

Cocommutative Hopf Algebra

S-matrix $\mathcal{S}$ permutes two particles (modules) $\mathbb{A}, \mathbb{B}$

$$\mathcal{S} : \mathbb{A} \otimes \mathbb{B} \to \mathbb{B} \otimes \mathbb{A}.$$

Can the S-matrix be invariant? Quasi-cocommutativity:

$$\mathcal{S} \circ \Delta(\mathfrak{J}^A) = \Delta(\mathfrak{J}^A) \circ \mathcal{S}.$$

Center: Matrix form of $\mathcal{S}$ irrelevant. Need $\mathfrak{P}_1 + \mathfrak{U}_1^2 \mathfrak{P}_2 = \mathfrak{P}_2 + \mathfrak{U}_2^2 \mathfrak{P}_1$.
Works in general only if central elements are identified [Plefka Spill Torrielli]

$$\mathfrak{P} = g\alpha^{+1}(1 - \mathfrak{U}^{+2}), \qquad \mathfrak{K} = g\alpha^{-1}(1 - \mathfrak{U}^{-2}).$$

Trade in two independent charges $\mathfrak{P}, \mathfrak{K}$ for one momentum charge $\mathfrak{U}$.
No constraint on energy $\mathfrak{C}$ (only through shortening $C^2 - PK = \frac{1}{4}$).

Fundamental S-Matrix

Invariance $\mathcal{S} \circ \Delta(\mathfrak{J}^A) = \Delta(\mathfrak{J}^A) \circ \mathcal{S}$ fixes S-matrix up to phase.
Tensor product of two fundamentals irreducible!

$$\langle p_1 \rangle_{\mathbf{4}} \otimes \langle p_2 \rangle_{\mathbf{4}} = \{0, 0; p_1, p_2\}_{\mathbf{16}} = \langle p_2 \rangle_{\mathbf{4}} \otimes \langle p_1 \rangle_{\mathbf{4}}$$

S-matrix equivalent to Shastry's R-matrix of 1D Hubbard model.[NB nlin.SI/0610017]

Consider **YBE** $\mathcal{S}_{12}\mathcal{S}_{13}\mathcal{S}_{23} = \mathcal{S}_{23}\mathcal{S}_{13}\mathcal{S}_{12}$. YBE involves tensor product

$$\langle p_1 \rangle_{\mathbf{4}} \otimes \langle p_2 \rangle_{\mathbf{4}} \otimes \langle p_3 \rangle_{\mathbf{4}} = \{0, 1; p_1, p_2, p_3\}_{\mathbf{32}} \oplus \{1, 0; p_1, p_2, p_3\}_{\mathbf{32}}.$$

Check for both components: $|\phi^1_1 \phi^1_2 \phi^1_3\rangle$, $|\psi^1_1 \psi^1_2 \psi^1_3\rangle$. Trivial!

Nevertheless, still have to do a little work to prove YBE.
Representation theory of full integrable symmetry should imply YBE.
Need larger symmetry: Yangian?!

Loop Algebras

Particle Models & Loop Algebras

Consider a generic integrable particle model.
Particles have flavour $\mathcal{A}$ and momentum p.

Understand flavour $\mathcal{A}$ as module of Lie algebra. What is p algebraically?
Answer: Evaluation parameter for a representation of a loop algebra.

The loop algebra (infinite-dimensional) $\{\mathfrak{J}^A_n\}$ of some Lie algebra $\{\mathfrak{J}^A\}$:

$$[\mathfrak{J}^A_n, \mathfrak{J}^B_m] = [\mathfrak{J}^A, \mathfrak{J}^B]_{n+m}.$$

Evaluation representations (finite-dimensional) defined as

$$\mathfrak{J}^A_n \, |\mathcal{A}, p\rangle = p^n \, \mathfrak{J}^A |\mathcal{A}, p\rangle.$$

Tensor products of evaluation representations are typically irreducible!
$\Longrightarrow$ S-matrix & YBE follow from representation theory.

S-Matrices & YBE

S-matrix $\mathcal{S}$ permutes two particles (modules) $\mathbb{A}, \mathbb{B}$

$$\mathcal{S} : \mathbb{A} \otimes \mathbb{B} \to \mathbb{B} \otimes \mathbb{A}.$$

- The tensor products $\mathbb{A} \otimes \mathbb{B}$ and $\mathbb{B} \otimes \mathbb{A}$ are irreducible.
- Quasi-cocommutativity: $\mathbb{A} \otimes \mathbb{B}$ and $\mathbb{B} \otimes \mathbb{A}$ are isomorphic.

S-matrix is the unique intertwiner (up to one overall factor: phase)

$$\mathcal{S} \circ \Delta(\mathfrak{J}_n^A) = \Delta(\mathfrak{J}_n^A) \circ \mathcal{S}.$$

Proof of Yang-Baxter equation:

$$\mathcal{S}_{\mathbb{A}\mathbb{B}}\mathcal{S}_{\mathbb{A}\mathbb{C}}\mathcal{S}_{\mathbb{B}\mathbb{C}} : \mathbb{A} \otimes \mathbb{B} \otimes \mathbb{C} \to \mathbb{C} \otimes \mathbb{B} \otimes \mathbb{A},$$

$$\mathcal{S}_{\mathbb{B}\mathbb{C}}\mathcal{S}_{\mathbb{A}\mathbb{C}}\mathcal{S}_{\mathbb{A}\mathbb{B}} : \mathbb{A} \otimes \mathbb{B} \otimes \mathbb{C} \to \mathbb{C} \otimes \mathbb{B} \otimes \mathbb{A}.$$

Map unique (up to overall factor) $\Longrightarrow$ YBE (almost).

Yangian Algebras

Loop algebra: Has trivial coproduct $\Delta(\mathfrak{J}^A_n) = \mathfrak{J}^A_n \otimes 1 + 1 \otimes \mathfrak{J}^A_n$.
Intertwiner $\mathbb{A} \otimes \mathbb{B} \to \mathbb{B} \otimes \mathbb{A}$ is trivial: $\mathcal{S} \sim \mathcal{P}$ (permutation operator).

Yangian: deformation of half of a loop algebra ($\mathfrak{J}^A_n, n \geq 0$).
Generated by $\mathfrak{J}^A = \mathfrak{J}^A_0$ and $\widehat{\mathfrak{J}}^A = \mathfrak{J}^A_1$. Coproduct:

$$\Delta(\mathfrak{J}^A) = \mathfrak{J}^A \otimes 1 + 1 \otimes \mathfrak{J}^A,$$
$$\Delta(\widehat{\mathfrak{J}}^A) = \widehat{\mathfrak{J}}^A \otimes 1 + 1 \otimes \widehat{\mathfrak{J}}^A + f^A_{BC}\, \mathfrak{J}^B \otimes \mathfrak{J}^C.$$

Coproduct now depends on the order of particles through $\mathfrak{J}^B \otimes \mathfrak{J}^C$.
Yangian has non-trivial S-matrix. Evaluation representations:

$$\widehat{\mathfrak{J}}^A |\mathcal{A}, p\rangle = (u(p) + u_0)\, \mathfrak{J}^A |\mathcal{A}, p\rangle, \qquad u(p) = \tfrac{1}{2}\cot(\tfrac{1}{2}p).$$

Double Yangian: deformation of full loop algebra.
Quantum-Deformed Affine Algebra: Similar, but also $\mathfrak{J}^A$ deformed.

Yangian

Yangian $\mathsf{Y}(\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3)$

Need to find a $\widehat{\mathfrak{J}}^A$ to enhance $\mathfrak{J}^A$ of $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$.
Coproduct of $\mathfrak{J}^A$ is braided by $\mathfrak{U}$

$$\Delta(\mathfrak{J}^A) = \mathfrak{J}^A \otimes 1 + \mathfrak{U}^{[A]} \otimes \mathfrak{J}^A.$$

Educated guess for braided coproduct of $\widehat{\mathfrak{J}}^A$ [NB arxiv:0704.0400]

$$\Delta(\widehat{\mathfrak{J}}^A) = \widehat{\mathfrak{J}}^A \otimes 1 + \mathfrak{U}^{[A]} \otimes \widehat{\mathfrak{J}}^A + f^A_{BC}\, \mathfrak{J}^B \mathfrak{U}^{[C]} \otimes \mathfrak{J}^C.$$

Coproduct of central elements

$$\Delta(\widehat{\mathfrak{C}}) = \widehat{\mathfrak{C}} \otimes 1 + 1 \otimes \widehat{\mathfrak{C}} + \mathfrak{P}\mathfrak{U}^{-2} \otimes \mathfrak{K} - \mathfrak{K}\mathfrak{U}^{+2} \otimes \mathfrak{P},$$
$$\Delta(\widehat{\mathfrak{P}}) = \widehat{\mathfrak{P}} \otimes 1 + \mathfrak{U}^{+2} \otimes \widehat{\mathfrak{P}} - \mathfrak{C}\mathfrak{U}^{+2} \otimes \mathfrak{P} + \mathfrak{P} \otimes \mathfrak{C},$$
$$\Delta(\widehat{\mathfrak{K}}) = \widehat{\mathfrak{K}} \otimes 1 + \mathfrak{U}^{-2} \otimes \widehat{\mathfrak{K}} + \mathfrak{C}\mathfrak{U}^{-2} \otimes \mathfrak{K} - \mathfrak{K} \otimes \mathfrak{C}.$$

Center is cocommutative if $\mathfrak{P}, \mathfrak{K}, \widehat{\mathfrak{P}}, \widehat{\mathfrak{K}} \sim (1 - \mathfrak{U}^{\pm 2})$!

Invariance of S-Matrix

Define evaluation representation

$$\widehat{\mathfrak{J}}^A|\mathcal{A},p\rangle = ig(u(p)+u_0)\mathfrak{J}^A|\mathcal{A},p\rangle.$$

Check invariance of S-matrix on two-particle states $|\mathcal{A},p\rangle \otimes |\mathcal{B},q\rangle$

$$\mathcal{S}\circ\Delta(\widehat{\mathfrak{J}}^A) = \Delta(\widehat{\mathfrak{J}}^A)\circ\mathcal{S}.$$

Satisfied if u related to momentum p (as previously assumed)

$$u = x^+ + \frac{1}{x^+} - \frac{i}{2g} = x^- + \frac{1}{x^-} + \frac{i}{2g}\,, \qquad e^{ip} = \frac{x^+}{x^-}\,.$$

S-matrix has Yangian symmetry. YBE follows.
More or less standard, but p already parameter of Lie representation.
Yangian for the one-dimensional Hubbard model.

Relation to Full Yangian

The $\mathfrak{psu}(2,2|4)$ Yangian looks different. [Dolan Nappi Witten] [Serban Staudacher] [Agarwal Rageev] [NB, Erkal in progress] [Zwiebel hep-th/0610283]

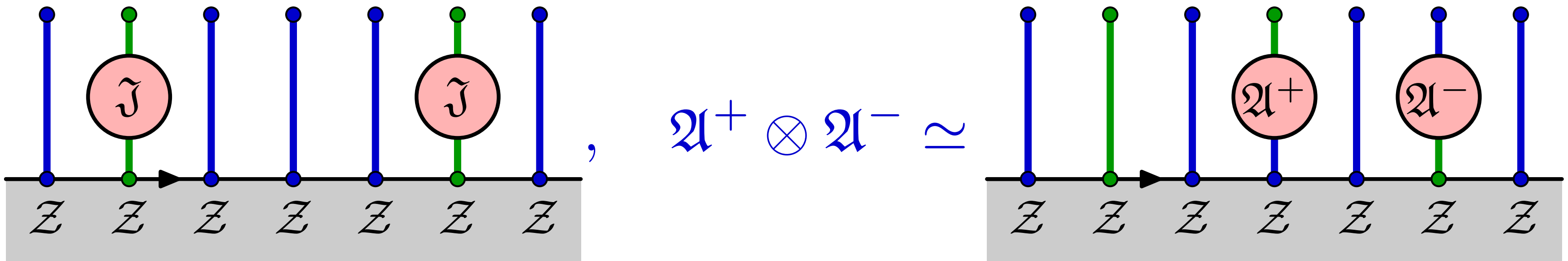
Let $\mathfrak{J} \in \mathfrak{psu}(2|2)^2$ and $\mathfrak{A}^{\pm} \perp \mathfrak{psu}(2|2)^2$. Then $\mathfrak{psu}(2,2|4)$ coproduct

$$\Delta\widehat{\mathfrak{J}} \simeq \widehat{\mathfrak{J}} \otimes 1 + 1 \otimes \widehat{\mathfrak{J}} + \mathfrak{J} \otimes \mathfrak{J} + \mathfrak{A}^{\pm} \otimes \mathfrak{A}^{\mp}.$$

How does the $\mathfrak{psu}(2|2)^2$ Yangian relate to the $\mathfrak{psu}(2,2|4)$ Yangian?

Action of full Yangian (bi-local only) on excitation states:

$\mathfrak{J} \otimes \mathfrak{J} \simeq$, $\mathfrak{A}^{+} \otimes \mathfrak{A}^{-} \simeq$.



Action of $\mathfrak{A}^{+} \otimes \mathfrak{A}^{-}$ should lead to non-trivial $\widehat{\mathfrak{J}} \sim u\mathfrak{J}$ on single excitations.

Constraints for the construction of the full Yangian?

Quantum Deformations

Quantum Deformations $\mathrm{U}_q(\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3)$

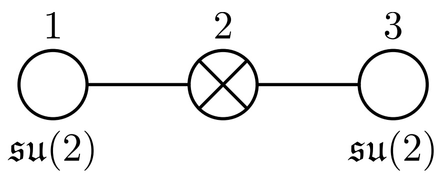
Quantum deformations of $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ [NB, Koroteev, Spill, in preparation]

- are presumably completely irrelevant for AdS/CFT,
- are nevertheless interesting mathematical and technical subject,
- may lead to a quantum deformation of the Hubbard model.

Use **Chevalley basis**: $\mathfrak{E}_k, \mathfrak{H}_k, \mathfrak{F}_k$, $k = 1, 2, 3$ (rank 3)

$$\begin{array}{llll}
\text{raising}: & \mathfrak{E}_1 \sim \mathfrak{R}^2{}_1, & \mathfrak{E}_2 \sim \mathfrak{Q}^2{}_2, & \mathfrak{E}_3 \sim \mathfrak{L}^1{}_2, \\
\text{Cartan}: & \mathfrak{H}_1 \sim 2\mathfrak{R}^2{}_2, & \mathfrak{H}_2 \sim -\mathfrak{C} - \mathfrak{R}^2{}_2 - \mathfrak{L}^2{}_2, & \mathfrak{H}_3 \sim 2\mathfrak{L}^2{}_2, \\
\text{lowering}: & \mathfrak{F}_1 \sim \mathfrak{R}^1{}_2, & \mathfrak{F}_2 \sim -\mathfrak{S}^2{}_2, & \mathfrak{F}_3 \sim \mathfrak{L}^2{}_1.
\end{array}$$

$\mathfrak{E}_k, \mathfrak{H}_k, \mathfrak{F}_k$ associated to node k of Dynkin diagram:



Quantum Deformed Algebra

Charges and commutators. (A_{jk}: symmetric Cartan matrix)

$$[\mathfrak{H}_j, \mathfrak{E}_k] = A_{jk}\mathfrak{E}_k, \quad [\mathfrak{E}_j, \mathfrak{F}_k] = \pm\delta_{jk}\frac{q^{\mathfrak{H}_j} - q^{-\mathfrak{H}_j}}{q - q^{-1}}, \quad \text{etc..}$$

Serre relations (relax two of them to obtain central extension)

$$\mathfrak{E}_1\mathfrak{E}_1\mathfrak{E}_2 - (q + q^{-1})\mathfrak{E}_1\mathfrak{E}_2\mathfrak{E}_1 + \mathfrak{E}_2\mathfrak{E}_1\mathfrak{E}_1 = 0, \quad \text{etc..}$$

Quantum deformed coproduct

$$\Delta(\mathfrak{H}_k) = \mathfrak{H}_k \otimes 1 + 1 \otimes \mathfrak{H}_k, \quad \Delta(\mathfrak{E}_k) = \mathfrak{E}_k \otimes 1 + q^{-\mathfrak{H}_k} \otimes \mathfrak{E}_k, \quad \text{etc..}$$

Braiding of $\mathfrak{E}_2, \mathfrak{F}_2$ as before; leads to cocommutativity of center.

$$\Delta(\mathfrak{E}_2) = \mathfrak{E}_2 \otimes 1 + q^{-\mathfrak{H}_k}\mathfrak{U}^{+1} \otimes \mathfrak{E}_2, \quad \text{etc..}$$

Fundamental Representation

Can construct a $(2|2)$-dimensional representation as before.

Find constraint (quantum deformed)

$$\left(\frac{q^C - q^{-C}}{q - q^{-1}}\right)^2 - PK = \left(\frac{q^{1/2} - q^{-1/2}}{q - q^{-1}}\right)^2.$$

Introduce $x^\pm$ parameters with genus-one constraint

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} + ig(q - q^{-1})\left(\frac{qx^+}{x^-} - \frac{x^-}{qx^+}\right) = \frac{i}{g}, \quad e^{ip} = \frac{qx^+}{x^-}.$$

S-matrix can be constructed. Scattering factor for alike bosons:

$$A_{12} = S^0_{12}\,\frac{q^{+1}x_2^+ - q^{-1}x_1^-}{q^{-1}x_2^- - q^{+1}x_1^+}.$$

Enhanced Symmetry in the $\mathfrak{psu}(1,1|2)$ Sector

Bethe Equations

$\mathfrak{psu}(1,1|2)$ sector comprises fields $\{\mathcal{D}^n\phi_{1,2}, \mathcal{D}^n\psi, \mathcal{D}^n\dot{\psi}\}$. Bethe equations

$$1 = \prod_{j=1}^{K} \frac{x_j^+}{x_j^-}, \qquad 1 = \prod_{j=1}^{K} \frac{y_k - x_j^+}{y_k - x_j^-}, \qquad 1 = \prod_{j=1}^{K} \frac{\dot{y}_k - x_j^+}{\dot{y}_k - x_j^-},$$

$$1 = \left(\frac{x_k^-}{x_k^+}\right)^L \prod_{\substack{j=1\\ j\neq k}}^{K} \left(\sigma_{12}^2 \frac{u_k - u_j + ig^{-1}}{u_k - u_j - ig^{-1}}\right) \prod_{j=1}^{N} \frac{x_k^- - y_j}{x_k^+ - y_j} \prod_{j=1}^{\dot{N}} \frac{x_k^- - \dot{y}_j}{x_k^+ - \dot{y}_j}.$$

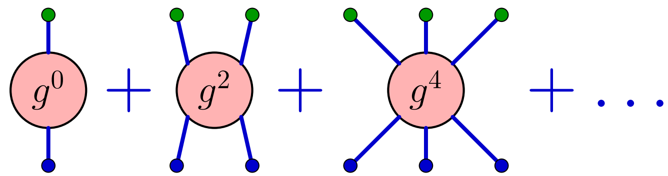
Symmetries: (modify set of Bethe roots; keep energy & charges)

- Add Bethe roots $x^{\pm}, y, \dot{y} = \infty$: $\mathfrak{psu}(1,1|2)$ manifest symmetry
- Add roots $y, \dot{y} = 0$, reduce length $L \mapsto L-1$: $\mathfrak{psu}(1|1)^2$ hidden sym.
- Change flavour between y and $\dot{y}$: 2^M degeneracy?! [NB, Staudacher hep-th/0504190]
  What is the symmetry origin of this degeneracy?

Symmetry Generators

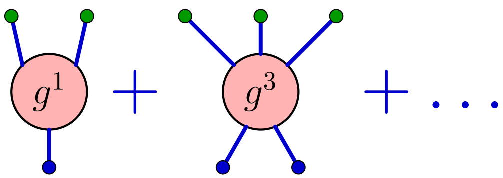
How are the symmetries realised as spin chain operators?

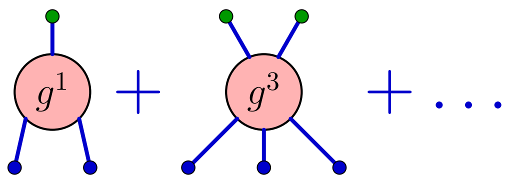
- $\mathfrak{psu}(1,1|2)$ preserves length. Expansion in even powers of g

$\mathfrak{J}(g) =$ 

  Action known at $\mathcal{O}(g^4)$. [Zwiebel hep-th/0511109]

- $\mathfrak{psu}(1|1)^2$ changes length by one unit. Expansion in odd powers of g

$\mathfrak{Q}(g) =$ 

$\mathfrak{S}(g) =$ 

  Action known at $\mathcal{O}(g^3)$. [Zwiebel hep-th/0511109]

- What about the 2^M degeneracy?

$\mathfrak{su}(2)$ Automorphism

The algebra $\mathfrak{psu}(1,1|2)$ has a $\mathfrak{su}(2)$ outer automorphism.

- Supercharges form $\mathfrak{su}(2)$ doublet $\mathfrak{Q}^{a\beta\mathfrak{c}} = (\varepsilon^{ad}\mathfrak{Q}^{\beta}{}_{c}, \varepsilon^{\beta\delta}\mathfrak{S}^{a}{}_{\delta})$.
- Fermions form doublet $\mathcal{D}^n\psi^{\mathfrak{a}} = (\mathcal{D}^n\psi, \mathcal{D}^n\dot{\psi})$.
- Bethe equations: Changes numbers $N, \dot{N}$ of auxiliary roots by ± 1.

Automorphism can be defined consistently for field representation [NB, Zwiebel to appear]

$$\mathfrak{B}^{\mathfrak{a}}{}_{\mathfrak{b}}|\mathcal{D}^n\phi^c\rangle = 0, \qquad \mathfrak{B}^{\mathfrak{a}}{}_{\mathfrak{b}}|\mathcal{D}^n\psi^{\mathfrak{c}}\rangle = \delta^{\mathfrak{c}}_{\mathfrak{b}}|\mathcal{D}^n\psi^{\mathfrak{a}}\rangle - \tfrac{1}{2}\delta^{\mathfrak{a}}_{\mathfrak{b}}|\mathcal{D}^n\psi^{\mathfrak{c}}\rangle.$$

- Automorphism explains some degeneracy:
  States organised into $\mathfrak{su}(2)$ multiplets of $\mathfrak{psu}(1,1|2)$ multiplets.
- Automorphism does not explain all degeneracy, e.g.: $\mathbf{2}^{\otimes 3} = \mathbf{4} \oplus \mathbf{2} \oplus \mathbf{2}$.
- $\mathfrak{su}(2)$ multiplets composed from linear combinations of Bethe states.

How do the additional hidden symmetry generators act?

Sample Degenerate States

Find some degenerate states to gain experience. [NB, Zwiebel to appear]

Simplest state which is part of a 2^{L-2} multiplet: $|0\rangle = |\psi^{<}\psi^{<}\ldots\psi^{<}\psi^{<}\rangle$.

Next simplest states should have three excitations. Basis states:

$$|j,k,\ell\rangle = \epsilon_{ab}|\ldots \overset{\overset{j}{\downarrow}}{\mathcal{D}\psi^{<}}\ldots \overset{\overset{k}{\downarrow}}{\phi^{a}}\ldots \overset{\overset{\ell}{\downarrow}}{\phi^{b}}\ldots\rangle.$$

Find $L+1$ degenerate states. Simplest one unrelated by symmetries:

$$|1\rangle \simeq \sum\nolimits_{j,k}(-1)^{k-j}|j,k,k+1\rangle.$$

Obtained by bi-local combination of $\mathfrak{psu}(1|1)^2$ generators

$$|1\rangle = \sum\nolimits_{j,k}\mathfrak{S}^{>}(j)\,\mathfrak{Q}^{>}(k)\,|0\rangle.$$

Caveat: Works unless state physical (zero momentum). Then: $\mathfrak{psu}(1|1)^2$.

Yangian Automorphism

Symmetry generated by

$$\widehat{\mathfrak{B}}^{\mathfrak{ab}} = \sum_{jk} \mathfrak{S}^{\mathfrak{a}}(j)\mathfrak{Q}^{\mathfrak{b}}(k)$$

- Commutes exactly with Hamiltonian!
- Generates the other degenerate states.
- Generates half of undeformed loop algebra of $\mathfrak{su}(2)$ automorphism (?)

What kind of generator is $\widehat{\mathfrak{B}}^{\mathfrak{ab}}$?

Part of $\mathfrak{psu}(1|1)^2$ Yangian algebra with automorphism?
Standard coproduct of $\widehat{\mathfrak{B}}^{\mathfrak{ab}}$ for this algebra

$$\Delta\widehat{\mathfrak{B}}^{\mathfrak{ab}} = \widehat{\mathfrak{B}}^{\mathfrak{ab}} \otimes 1 + 1 \otimes \widehat{\mathfrak{B}}^{\mathfrak{ab}} + \mathfrak{Q}^{\mathfrak{a}} \otimes \mathfrak{S}^{\mathfrak{b}} + \mathfrak{S}^{\mathfrak{a}} \otimes \mathfrak{Q}^{\mathfrak{b}}.$$

Bi-local part similar to above definition: Part of Yangian! Strange:
$\widehat{\mathfrak{B}}^{\mathfrak{ab}}$ non-trivial for $e^{iP} \neq 1$, but $\mathfrak{psu}(1|1)^2$ only when $e^{iP} = 1$.

Conclusions

Conclusions

⋆ **Extended $\mathfrak{psl}(2|2)$ Algebra**

- Extensions of $\mathfrak{psl}(2|2)$ are interesting algebras.
- Applications to AdS/CFT and one-dimensional Hubbard model.
- Quantum deformation $\mathrm{U}_q(\mathfrak{psl}(2|2))$ possible.

⋆ **Loop Algebras and Deformations**

- Integrable models governed by infinite-dimensional Hopf algebras.
- S-matrix has Yangian symmetry.
- Degeneracy in $\mathfrak{psu}(1,1|2)$ sector explained by Yangian.

⋆ **Outlook**

- Apply symmetries to understand the AdS/CFT integrable system.