The 12th Claude Itzykson Meeting
Integrability in Gauge and String Theory
Saclay, 19 June 2007

TBA?

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School of Mathematics
Trinity College Dublin

work in progress with Gleb Arutyunov and Marija Zamaklar

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Thermodynamic Bethe Ansatz?

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Introduction

- IIB strings in $AdS_5 \times S^5 \equiv \mathcal{N} = 4\ SU(N)$ SYM — Maldacena '97

  Spectrum of strings on $AdS_5 \times S^5$ $\equiv$ Spectrum of scaling dimensions of SYM operators

- Green-Schwarz string in $AdS_5 \times S^5$ — Metsaev,Tseytlin '98

  in light-cone gauge — Frolov, Plefka, Zamaklar '06

  – is a 2d model on a cylinder of circumference $P_+ = (1-a)J + aE$

  – has massive spectrum

- Decompactifying limit: λ fixed, $P_+ \to \infty$ [Ambjorn,Janik,Kristjansen '05]; [Klose,Zarembo '06]; [Janik '06]; [Arutyunov, Frolov '06]; [Hofman, Maldacena '06]

  – 2d integrable model on a plane with massive excitations

  – Asymptotic states and S-matrix are well-defined

  – Model is not Lorentz invariant

Introduction

- (Super)-charges commuting with the l.c. Hamiltonian form the centrally-extended $su(2|2) \oplus su(2|2)$ algebra. The central charges are proportional to

$$\sin \frac{\Delta x_-}{2} = \sin \frac{p_{\rm ws}}{2}$$

and vanish on physical subspace — Arutyunov, Frolov, Plefka, Zamaklar '06

- The same algebra appears in the gauge theory spin chain — Beisert '05
- Spin chain dispersion relation: — Beisert,Dippel,Staudacher '04

$$\mathcal{E}(p) = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}}$$

- S-matrix is determined by symmetries up to a scalar factor — Beisert '05

Introduction: Dressing Phase

Gauge-independent dressing phase — Arutyunov, Frolov, Staudacher '04

$$\theta(p_1,p_2)=\sum_{r=2}^{\infty}\sum_{n=0}^{\infty}c_{r,r+1+2n}(\lambda)\left[q_r(p_1)q_{r+1+2n}(p_2)-q_r(p_2)q_{r+1+2n}(p_1)\right]$$

- The dressing phase is believed to be known up to the first two orders in expansion in $1/\sqrt{\lambda}$ — Arutyunov, Frolov, Staudacher '04; Hernandez, Lopez '06
- It is conjectured to satisfy crossing-type relations — Janik '06
- Janik's equations were checked up to the first two orders in expansion in $1/\sqrt{\lambda}$ — Arutyunov, Frolov '06
- All loop expansion in $1/\sqrt{\lambda}$ was conjectured — Beisert, Hernandez, Lopez '06
- All loop expansion in powers of λ was conjectured — Beisert, Eden, Staudacher '06
- It agrees with 4-loop computation — Bern, Czakon, Dixon, Kosower, Smirnov '06

Introduction: TBA

Let's assume the dressing phase is known.

What's next? Understand the spectrum at finite $P_+ = (1-a)J + aE$ or J.

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- The vanishing of the finite J corrections to the vacuum state should impose additional restrictions on the dressing factor, and, therefore, would provide a nontrivial test of the conjectured dressing factor

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- The vanishing of the finite J corrections to the vacuum state should impose additional restrictions on the dressing factor, and, therefore, would provide a nontrivial test of the conjectured dressing factor
- The TBA equations may give an idea how the finite J Bethe equations should look like [P.Dorey,Tateo '96]; [Teschner '07]

Introduction: TBA

What steps are required?

- Find the dispersion relation, the S-matrix, and Bethe equations for the mirror model. The thermodynamic limit is taken for the mirror system.

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- Analyze the structure of solutions of these Bethe equations in the large R limit, and formulate some kind of a string hypotheses. The first step is to analyze bound states of the mirror model

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What steps are required?

- Find the dispersion relation, the S-matrix, and Bethe equations for the mirror model. The thermodynamic limit is taken for the mirror system.
- Analyze the structure of solutions of these Bethe equations in the large R limit, and formulate some kind of a string hypotheses. The first step is to analyze bound states of the mirror model
- Take the thermodynamic limit, and derive the integral Bethe equations for densities of strings and states
- Count the states and compute the entropy in the thermodynamic limit
- Minimize the free energy, and derive the TBA equations

Outline

- Double-Wick rotation, mirror Hamiltonian and TBA
- Mirror dispersion relation
- Mirror S-matrix and supersymmetry algebra
- Crossing symmetry and physical unitarity
- Double-Wick rotation and the generalized rapidity
- BAE and bound states
- Conclusion and open problems

Double-Wick rotation, mirror Hamiltonian and TBA

We follow the approach developed by Zamolodchikov '90

Consider 2-dim field theory defined on a circle of circumference L:

$$H = \int_0^L d\sigma\, \mathcal{H}(p, x, x')$$

It does not have to be relativistic invariant.

Partition function of the model

$$Z(R, L) \equiv \sum_n \langle\psi_n|e^{-HR}|\psi_n\rangle = \sum_n e^{-E_n R}\,,$$

where $|\psi_n\rangle$ is the complete set of eigenvectors of H.

By using the standard path integral representation, we get

$$Z(R, L) = \int \mathcal{D}p\, \mathcal{D}x\, e^{\int_0^R d\tau\, \int_0^L\, d\sigma(ip\dot{x} - \mathcal{H})}\,,$$

where the integration is taken over x and p periodic in both τ and σ.

Double-Wick rotation, mirror Hamiltonian and TBA

$$Z(R,L) = \int \mathcal{D}p\,\mathcal{D}x\, e^{\int_0^R d\tau \int_0^L \, d\sigma(ip\dot{x} - \mathcal{H})}$$

- $-\int_0^R d\tau \int_0^L \, d\sigma(ip\dot{x} - \mathcal{H})$ is Euclidean action.
- Integrating over p with $\int_0^R d\tau \int_0^L \, d\sigma(p\dot{x} - \mathcal{H}) \;\Rightarrow$ Minkowski action.
- Euclidean action: replace $\dot{x} \to i\dot{x} \;\Leftrightarrow$ Wick rotation $\tau \to -i\tau$.

Double-Wick rotation, mirror Hamiltonian and TBA

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Take the Euclidean action

- Replace $x' \rightarrow -ix' \;\Leftrightarrow$ Wick rotation of the σ-coordinate $\sigma \rightarrow i\sigma$.
- New action where σ is considered as the new time coordinate.
- Let $\widetilde{H}$ be the Hamiltonian with respect to σ

$$\widetilde{H} = \int_0^R d\tau\, \widetilde{\mathcal{H}}(\widetilde{p}, x, \dot{x})\,,$$

$\widetilde{p}$ are canonical momenta of the coordinates x with respect to σ.

Double-Wick rotation, mirror Hamiltonian and TBA

We refer to the model with the Hamiltonian $\widetilde{H}$ as to the mirror theory.

No Lorentz invariance $\Rightarrow$ H and $\widetilde{H}$ describe *different* Minkowski theories.

The partition function of the mirror model

$$\widetilde{Z}(R,L) \equiv \sum_n \langle \widetilde{\psi}_n | e^{-\widetilde{H}L} | \widetilde{\psi}_n \rangle = \sum_n e^{-\widetilde{E}_n L} = \int \mathcal{D}\widetilde{p}\, \mathcal{D}x\, e^{\int_0^R d\tau \int_0^L d\sigma (i\widetilde{p}x' - \widetilde{\mathcal{H}})}$$

Integrating over $\widetilde{p}$, we get the same Euclidean action $\Rightarrow$

$$\widetilde{Z}(R,L) = Z(R,L)\,.$$

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Integrating over $\widetilde{p}$, we get the same Euclidean action $\Rightarrow$

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Take the limit $R \to \infty$

- $\log Z(R,L) \sim -RE(L)$, where $E(L)$ is the ground state energy.
- $\log \widetilde{Z}(R,L) \sim -RLf(L)$, where $f(L)$ is the bulk free energy of the mirror theory at the temperature $T = 1/L$.

Double-Wick rotation, mirror Hamiltonian and TBA

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- $\log \widetilde{Z}(R,L) \sim -RLf(L)$, where $f(L)$ is the bulk free energy of the mirror theory at the temperature $T = 1/L$.
- This leads to the relation

$$\boxed{E(L) = Lf(L)}$$

To find the free energy we can use the thermodynamic Bethe ansatz because $R >> 1$. This requires

- The mirror S-matrix
- The asymptotic Bethe equations for the mirror system
- The l.c. string theory is not Lorentz invariant $\Rightarrow$ $\widetilde{H} \neq H$.

Mirror dispersion relation

Dispersion relation from the pole structure of 2-point correlation function.

Compute it in Euclidean space. Both dispersion relations from

$$\mathcal{E}_{\rm E}^2 + \frac{\lambda}{\pi^2}\sin^2\frac{p_{\rm E}}{2} + 1$$

which appears in the pole of the 2-point correlation function.

Usual dispersion relation follows from the analytic continuation

$$\mathcal{E}_{\rm E} \to -i\mathcal{E}\,, \quad p_{\rm E} \to p \Rightarrow \boxed{\mathcal{E}^2 = \frac{\lambda}{\pi^2}\sin^2\frac{p}{2} + 1}\,,$$

and the mirror one from

$$\mathcal{E}_{\rm E} \to \widetilde{p}\,, \quad p_{\rm E} \to i\widetilde{\mathcal{E}} \Rightarrow \boxed{\widetilde{\mathcal{E}} = 2\,{\rm arcsinh}\left(\frac{\pi}{\sqrt{\lambda}}\sqrt{1+\widetilde{p}^2}\right)}\,.$$

Mirror dispersion relation

$$\mathcal{E}_{\rm E} \to -i\mathcal{E}\,, \quad p_{\rm E} \to p \;\Rightarrow\; \mathcal{E}^2 = \frac{\lambda}{\pi^2}\sin^2\frac{p}{2} + 1\,,$$

$$\mathcal{E}_{\rm E} \to \widetilde{p}\,, \quad p_{\rm E} \to i\widetilde{\mathcal{E}} \;\Rightarrow\; \widetilde{\mathcal{E}} = 2\,\mathrm{arcsinh}\left(\frac{\pi}{\sqrt{\lambda}}\sqrt{1+\widetilde{p}^2}\right)\,.$$

Thus, p and $\widetilde{p}$ are related by the analytic continuation

$$\boxed{p \to 2i\,\mathrm{arcsinh}\left(\frac{\pi}{\sqrt{\lambda}}\sqrt{1+\widetilde{p}^2}\right)}\,, \qquad \boxed{\mathcal{E} = \sqrt{\frac{\lambda}{\pi^2}\sin^2\frac{p}{2} + 1} \to i\widetilde{p}}$$

Plane-wave type limit in the mirror theory: $\lambda \to \infty$ with $\widetilde{p}$ fixed

$$\widetilde{\mathcal{E}}_{\rm ppw} = \frac{2\pi}{\sqrt{\lambda}}\sqrt{1+\widetilde{p}^2}\,,$$

Semi-classical limit in the mirror theory: $\lambda \to \infty$ with $\widetilde{p}/\sqrt{\lambda}$ fixed

$$\widetilde{\mathcal{E}}_{\rm sc} = 2\,\mathrm{arcsinh}\left(\frac{\pi|\widetilde{p}|}{\sqrt{\lambda}}\right)\,.$$

Mirror giant magnon?

Hofman, Maldacena '06

Mirror giant magnon

is a one-soliton solution of the mirror model which plays the same role as the HM giant magnon solution.

- $\mathrm{AdS}_5 \times \mathrm{S}^5$ string theory: keep one S^5 field in the light-cone Hamiltonian

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- $\mathrm{AdS}_5 \times \mathrm{S}^5$ string theory: keep one S^5 field in the light-cone Hamiltonian
- Mirror model: there are no solitons if one considers the S^5 fields in $\widetilde{H}$

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- Mirror model: keep one AdS_5 field in the mirror Hamiltonian

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- Mirror model: there are no solitons if one considers the S^5 fields in $\widetilde{H}$
- Mirror model: keep one AdS_5 field in the mirror Hamiltonian
- Ansatz for the mirror magnon is Arutyunov, Frolov, Zamaklar '06

$$z = z(\tilde{\sigma} - \tilde{v}\tilde{\tau})$$

- Mirror magnon dispersion relation

$$\widetilde{\mathcal{E}}_{\mathrm{mm}} = 2\,\mathrm{arcsinh}\left(\frac{\pi|\widetilde{p}|}{\sqrt{\lambda}}\right)$$

- Roles of S^5 and AdS_5 are exchanged in mirror models

Mirror S-matrix and supersymmetry algebra

S-matrix from 4-point correlation functions by using the LSZ reduction

The mirror S-matrix is related to the original one by analytic continuation

$$\widetilde{S}(\widetilde{p}_1, \widetilde{p}_2) = S(p_1, p_2)\,, \quad p_k \to 2i\,\mathrm{arcsinh}\left(\frac{\pi}{\sqrt{\lambda}}\sqrt{1+\widetilde{p}_k^2}\right)$$

For real $\widetilde{p}_k$ the resulting mirror S-matrix $\widetilde{S}$ should

- satisfy the YB equation, unitarity, physical unitarity, crossing relation
- be $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ invariant

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The states of the mirror theory also should carry unitary representations of $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$.

Is there a way to implement the double-Wick rotation on the symmetry algebra level?

Centrally-extended $su(2|2)$ algebra

$$\mathfrak{su}(2|2)_{H,C} \ni \{\mathbf{L}_a{}^b\,,\ \mathbf{R}_\alpha{}^\beta \in \mathfrak{su}(2)\oplus\mathfrak{su}(2);\, \mathbf{Q}_\alpha{}^a,\ \ \mathbf{Q}^{\dagger\,\alpha}_a;\, \mathbf{H},\, \mathbf{C},\, \mathbf{C}^\dagger\}$$

$$\Big[\mathbf{L}_a{}^b, \mathbf{J}_c\Big] = \delta_c^b \mathbf{J}_a - \frac{1}{2}\delta_a^b \mathbf{J}_c\,, \qquad \Big[\mathbf{R}_\alpha{}^\beta, \mathbf{J}_\gamma\Big] = \delta_\gamma^\beta \mathbf{J}_\alpha - \frac{1}{2}\delta_\alpha^\beta \mathbf{J}_\gamma\,,$$

$$\Big[\mathbf{L}_a{}^b, \mathbf{J}^c\Big] = -\delta_a^c \mathbf{J}^b + \frac{1}{2}\delta_a^b \mathbf{J}^c\,, \qquad \Big[\mathbf{R}_\alpha{}^\beta, \mathbf{J}^\gamma\Big] = -\delta_\alpha^\gamma \mathbf{J}^\beta + \frac{1}{2}\delta_\alpha^\beta \mathbf{J}^\gamma\,,$$

$$\{\mathbf{Q}_\alpha{}^a, \mathbf{Q}^{\dagger\,\beta}_b\} = \delta_b^a \mathbf{R}_\alpha{}^\beta + \delta_\alpha^\beta \mathbf{L}_b{}^a + \frac{1}{2}\delta_b^a \delta_\alpha^\beta\ \mathbf{H}\,,$$

$$\{\mathbf{Q}_\alpha{}^a, \mathbf{Q}_\beta{}^b\} = \epsilon_{\alpha\beta}\epsilon^{ab}\ \mathbf{C}\,, \qquad \{\mathbf{Q}^{\dagger\,\alpha}_a, \mathbf{Q}^{\dagger\,\beta}_b\} = \epsilon_{ab}\epsilon^{\alpha\beta}\ \mathbf{C}^\dagger$$

- $\mathbf{H}$ is the gauge-fixed string Hamiltonian
- $\mathbf{C} = \frac{i}{2} g\,(e^{i\mathbf{P}} - 1)e^{2i\xi}\,, \quad g = \frac{\sqrt{\lambda}}{2\pi} = \frac{R^2}{2\pi\alpha'}$ Arutyunov, Frolov, Plefka, Zamaklar '06
- The phase ξ reflects the $\mathrm{U}(1)$ automorphism: $\mathbf{Q} \to e^{i\xi}\mathbf{Q}\,,\ \mathbf{C} \to e^{2i\xi}\mathbf{C}$
- Here we fix $\xi = -\frac{1}{4}\mathbf{P} \ \Rightarrow$ $\boxed{\mathbf{C} = \mathbf{C}^\dagger = -g\sin\frac{\mathbf{P}}{2}}$

Change of generators

Symmetry algebra of the mirror theory is a different real slice of the complexified $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ algebra.

Consider the following linear change of the generators

$$\widetilde{\mathbf{Q}}_\alpha{}^a = \frac{1}{\sqrt{2}}\left(\mathbf{Q}_\alpha{}^a + i\,\epsilon^{ac}\,\mathbf{Q}^{\dagger\,\gamma}_c\,\epsilon_{\gamma\alpha}\right)\,, \quad \widetilde{\mathbf{Q}}^{\dagger\,\alpha}_a = \frac{1}{\sqrt{2}}\left(\mathbf{Q}^{\dagger\,\alpha}_a + i\,\epsilon^{\alpha\beta}\,\mathbf{Q}_\beta{}^b\,\epsilon_{ba}\right)$$

Then, we find

$$\{\widetilde{\mathbf{Q}}_\alpha{}^a, \widetilde{\mathbf{Q}}^{\dagger\,\beta}_b\} = \delta^a_b \mathbf{R}_\alpha{}^\beta + \delta^\beta_\alpha \mathbf{L}_b{}^a - \frac{1}{2}\delta^a_b \delta^\beta_\alpha\, 2ig\sin\frac{\mathbf{P}}{2}\,,$$

$$\{\widetilde{\mathbf{Q}}_\alpha{}^a, \widetilde{\mathbf{Q}}_\beta{}^b\} = -\epsilon_{\alpha\beta}\epsilon^{ab}\,\frac{i}{2}\mathbf{H}\,, \quad \{\widetilde{\mathbf{Q}}^{\dagger\,\alpha}_a, \widetilde{\mathbf{Q}}^{\dagger\,\beta}_b\} = -\epsilon_{ab}\epsilon^{\alpha\beta}\,\frac{i}{2}\mathbf{H}$$

Perform the analytic continuation

$$\mathbf{P} \to i\widetilde{\mathbf{H}}\,, \quad \mathbf{H} \to i\widetilde{\mathbf{P}}$$

Mirror algebra

and obtain the mirror algebra

$$\{\widetilde{\mathbf{Q}}_\alpha{}^a, \widetilde{\mathbf{Q}}^{\dagger\,\beta}_b\} = \delta^a_b \mathbf{R}_\alpha{}^\beta + \delta^\beta_\alpha \mathbf{L}_b{}^a + g\,\delta^a_b\delta^\beta_\alpha\,\sinh\frac{\widetilde{\mathbf{H}}}{2}\,,$$

$$\{\widetilde{\mathbf{Q}}_\alpha{}^a, \widetilde{\mathbf{Q}}_\beta{}^b\} = \epsilon_{\alpha\beta}\epsilon^{ab}\,\frac{\widetilde{\mathbf{P}}}{2}\,, \quad \{\widetilde{\mathbf{Q}}^{\dagger\,\alpha}_a, \widetilde{\mathbf{Q}}^{\dagger\,\beta}_b\} = \epsilon_{ab}\epsilon^{\alpha\beta}\,\frac{\widetilde{\mathbf{P}}}{2}$$

- We can impose: $\left(\widetilde{\mathbf{Q}}_\alpha{}^a\right)^\dagger = \widetilde{\mathbf{Q}}^{\dagger\,\alpha}_a$ and $\widetilde{\mathbf{H}}^\dagger = \widetilde{\mathbf{H}}\,,\ \widetilde{\mathbf{P}}^\dagger = \widetilde{\mathbf{P}}$
- This algebra leads to the mirror dispersion relation:

$$4g^2\text{arcsinh}^2\frac{\widetilde{\mathcal{E}}}{2} = 1 + \widetilde{p}^2$$

- The S-matrix with $\mathbf{C} = \mathbf{C}^\dagger$ coincides with the string S-matrix
- The string S-matrix depends on η's which were partially fixed by requiring that the S-matrix satisfied YBE. Arutyunov,Frolov,Zamaklar '06

Fundamental representation

- The string S-matrix depends on η's which were partially fixed by requiring that the S-matrix satisfied YBE. Arutyunov,Frolov,Zamaklar '06

- The requirement that the representation remains unitary after the analytic continuation fixes the parameters η's basically uniquely.

- We get the fundamental rep of supersymmetry generators

$$\mathbf{Q}_\alpha{}^a|e_b\rangle = |e_M\rangle Q^{aM}_{\alpha b} = a\,\delta^a_b|e_\alpha\rangle\,, \qquad \mathbf{Q}_\alpha{}^a|e_\beta\rangle = b\,\epsilon_{\alpha\beta}\epsilon^{ab}|e_b\rangle\,,$$
$$\mathbf{Q}^{\dagger\,\alpha}_a|e_\beta\rangle = |e_M\rangle\overline{Q}^{\alpha M}_{a\beta} = d\,\delta^\alpha_\beta|e_a\rangle\,, \qquad \mathbf{Q}^{\dagger\,\alpha}_a|e_b\rangle = c\,\epsilon_{ab}\epsilon^{\alpha\beta}|e_\beta\rangle$$

where $d = a\,,\ c = b$, and

$$a = \sqrt{\frac{igx^- - igx^+}{2}} = \sqrt{\frac{H+1}{2}}\,, \quad b = -\sqrt{\frac{ig}{2x^+} - \frac{ig}{2x^-}} = -\sqrt{\frac{H-1}{2}}$$

- Both the original and mirror (analytically-continued) representations are unitary with respect to their own reality conditions.

Hopf algebra

$$a = \sqrt{\frac{igx^- - igx^+}{2}} = \sqrt{\frac{H+1}{2}}\,, \quad b = -\sqrt{\frac{ig}{2x^+} - \frac{ig}{2x^-}} = -\sqrt{\frac{H-1}{2}}$$

- These formulas define the action of the algebra generators of the original and mirror theories on one-particle states
- We need to know their action on multi-particle states
- Hopf algebra structure of $\mathcal{A}$ generated by $\mathbf{L}_a{}^b\,,\ \mathbf{R}_\alpha{}^\beta$, and $\mathbf{Q}_\alpha{}^a\,,\ \mathbf{Q}^{\dagger\,\alpha}_a$ and $\mathbf{H},\ \mathbf{P}$ subject to the algebra relations with $\mathbf{C} = \mathbf{C}^\dagger = -g\sin\frac{\mathbf{P}}{2}$
- We use the Hopf algebra introduced by Arutyunov,Frolov,Zamaklar '06

  which is basically equivalent to the Hopf algebras discussed in

  [Gomez,Hernandez '06]; [Plefka,Spill,Torrielli '06]

Hopf algebra

Hopf algebra structure of $\mathcal{A}$ generated by $\mathbf{L}_a{}^b$, $\mathbf{R}_\alpha{}^\beta$, and $\mathbf{Q}_\alpha{}^a$, $\mathbf{Q}^{\dagger\,\alpha}_a$ and $\mathbf{H}$, $\mathbf{P}$ subject to the algebra relations with $\mathbf{C} = \mathbf{C}^\dagger = -g \sin \frac{\mathbf{P}}{2}$

- co-product

$$
\begin{aligned}
\Delta(\mathbf{J}) &= \mathbf{J} \otimes \mathrm{id} + \mathrm{id} \otimes \mathbf{J} \quad \text{for any even generator}\,, \\
\Delta(\mathbf{Q}_\alpha{}^a) &= \mathbf{Q}_\alpha{}^a \otimes e^{i\mathbf{P}/4} + e^{-i\mathbf{P}/4} \otimes \mathbf{Q}_\alpha{}^a\,, \\
\Delta(\mathbf{Q}^{\dagger\,\alpha}_a) &= \mathbf{Q}^{\dagger\,\alpha}_a \otimes e^{-i\mathbf{P}/4} + e^{i\mathbf{P}/4} \otimes \mathbf{Q}^{\dagger\,\alpha}_a\,.
\end{aligned}
$$

  - Graded tensor product: for any algebra elements a, b, c, d

$$
(a \otimes b)(c \otimes d) = (-1)^{\epsilon(b)\epsilon(c)}(ac \otimes bd),
$$

    where $\epsilon(a) = 0$ if a is even, and $\epsilon(b) = -1$ if a is odd.

  - The co-product is compatible with the hermiticity conditions

Hopf algebra

Hopf algebra structure of $\mathcal{A}$ generated by $\mathbf{L}_a{}^b$, $\mathbf{R}_\alpha{}^\beta$, and $\mathbf{Q}_\alpha{}^a$, $\mathbf{Q}^{\dagger\,\alpha}_a$ and $\mathbf{H}$, $\mathbf{P}$ subject to the algebra relations with $\mathbf{C} = \mathbf{C}^\dagger = -g \sin\frac{\mathbf{P}}{2}$

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\Delta(\mathbf{Q}_\alpha{}^a) &= \mathbf{Q}_\alpha{}^a \otimes e^{i\mathbf{P}/4} + e^{-i\mathbf{P}/4} \otimes \mathbf{Q}_\alpha{}^a\,, \\
\Delta(\mathbf{Q}^{\dagger\,\alpha}_a) &= \mathbf{Q}^{\dagger\,\alpha}_a \otimes e^{-i\mathbf{P}/4} + e^{i\mathbf{P}/4} \otimes \mathbf{Q}^{\dagger\,\alpha}_a\,.
\end{aligned}
$$

  – Graded tensor product: for any algebra elements a, b, c, d

$$
(a \otimes b)(c \otimes d) = (-1)^{\epsilon(b)\epsilon(c)}(ac \otimes bd),
$$

  where $\epsilon(a) = 0$ if a is even, and $\epsilon(b) = -1$ if a is odd.

  – The co-product is compatible with the hermiticity conditions

- Antipode, S, is trivial for *any* algebra element

$$
S(\mathbf{J}) = -\mathbf{J}\,, \quad S(\mathbf{Q}) = -\mathbf{Q}\,, \quad S(\mathbf{Q}^\dagger) = -\mathbf{Q}^\dagger
$$

Hopf algebra

Co-product action on the supersymmetry generators $\widetilde{\mathbf{Q}}\,,\widetilde{\mathbf{Q}}^\dagger$

$$
\begin{aligned}
\Delta(\widetilde{\mathbf{Q}}_\alpha{}^a) &= \widetilde{\mathbf{Q}}_\alpha{}^a \otimes \cosh\left(\frac{\widetilde{\mathbf{H}}}{4}\right) + \cosh\left(\frac{\widetilde{\mathbf{H}}}{4}\right) \otimes \widetilde{\mathbf{Q}}_\alpha{}^a \\
&\quad - i\epsilon^{ad}\widetilde{\mathbf{Q}}^{\dagger\,\delta}_d \epsilon_{\delta\alpha} \otimes \sinh\left(\frac{\widetilde{\mathbf{H}}}{4}\right) + i\sinh\left(\frac{\widetilde{\mathbf{H}}}{4}\right) \otimes \epsilon^{ad}\widetilde{\mathbf{Q}}^{\dagger\,\delta}_d \epsilon_{\delta\alpha}\,, \\
\Delta(\widetilde{\mathbf{Q}}^{\dagger\,\alpha}_a) &= \widetilde{\mathbf{Q}}^{\dagger\,\alpha}_a \otimes \cosh\left(\frac{\widetilde{\mathbf{H}}}{4}\right) + \cosh\left(\frac{\widetilde{\mathbf{H}}}{4}\right) \otimes \widetilde{\mathbf{Q}}^{\dagger\,\alpha}_a \\
&\quad + i\epsilon^{\alpha\delta}\widetilde{\mathbf{Q}}_\delta{}^d \epsilon_{da} \otimes \sinh\left(\frac{\widetilde{\mathbf{H}}}{4}\right) - i\sinh\left(\frac{\widetilde{\mathbf{H}}}{4}\right) \otimes \epsilon^{\alpha\delta}\widetilde{\mathbf{Q}}_\delta{}^d \epsilon_{da}\,.
\end{aligned}
$$

- It is compatible with the hermiticity conditions of the mirror theory
- $\mathfrak{su}(2|2)$-invariant S-matrix can be always chosen to be unitary.
- What about the scalar factor? Physical unitarity of the mirror S-matrix follows from the crossing relations.

Crossing and Unitarity

Repeating the computation in Arutyunov,Frolov,Zamaklar '06

we find that the crossing relations are not modified.

In the $a = 0$ light-cone gauge the scalar factor of the string S-matrix is

$$S_0(x_1^\pm, x_2^\pm)^2 = s(x_1^\pm, x_2^\pm)\, e^{i\theta(x_1^\pm, x_2^\pm)}\,, \quad s(x_1^\pm, x_2^\pm) = \frac{x_2^+ - x_1^-}{x_1^+ - x_2^-}\, \frac{1 - \frac{1}{x_1^+ x_2^-}}{1 - \frac{1}{x_1^- x_2^+}}$$

Observe that if we write θ_{12} in terms of functions $\chi^{\pm,\pm}$ Arutyunov,Frolov '06

$$\theta(x_1^+, x_1^-, x_2^+, x_2^-) = \theta(x_1^-, x_1^+, x_2^-, x_2^+)$$

Thus, the complex conjugate θ^* for the mirror theory is

$$\theta_{12}^*(x_1^+, x_1^-, x_2^+, x_2^-) \equiv \theta_{12}(1/x_1^-, 1/x_1^+, 1/x_2^-, 1/x_2^+) =$$
$$\theta_{12}(1/x_1^+, 1/x_1^-, 1/x_2^+, 1/x_2^-) = \theta_{\bar{1}\bar{2}}(x_1^+, x_1^-, x_2^+, x_2^-)$$

Crossing and Unitarity

$$\theta_{12}^* = \theta_{\bar{1}\bar{2}} \;\Rightarrow\; \sigma_{12}^* = \frac{1}{\sigma_{\bar{1}\bar{2}}}\,, \qquad \sigma_{12} \equiv e^{i\theta_{12}}$$

Next we use that as follows from the crossing relations

$$S_0(1,2)^2 = S_0(\bar{1},\bar{2})^2 \quad \Rightarrow \quad \sigma(\bar{1},\bar{2}) = \frac{s(1,2)}{s(\bar{1},\bar{2})}\sigma(1,2) \quad \Rightarrow$$

$$\boxed{\sigma(1,2)^* = \frac{s(\bar{1},\bar{2})}{s(1,2)}\,\frac{1}{\sigma(1,2)}}\,, \quad \theta_{12}^* = \theta_{12} - i\ln\left(\frac{s(1,2)}{s(\bar{1},\bar{2})}\right)$$

Finally we need the following identities

$$\sigma(1,2)^* = \frac{1}{\sigma(\bar{1},\bar{2})}\,, \quad s(1,2)^* = \left(\frac{x_1^-}{x_1^+}\,\frac{x_2^+}{x_2^-}\right)^2 \frac{1}{s(1,2)}$$

$$\boxed{s(\bar{1},\bar{2}) = \left(\frac{x_1^+}{x_1^-}\,\frac{x_2^-}{x_2^+}\right)^2 s(1,2)} \;\Rightarrow\; \boxed{\sigma(1,2)^* = \left(\frac{x_1^+}{x_1^-}\,\frac{x_2^-}{x_2^+}\right)^2 \frac{1}{\sigma(1,2)}}$$

Crossing and Unitarity

With the help of these expressions we get

$$\left(S_0(1,2)^2\right)^* = \left(\frac{x_1^-}{x_1^+}\frac{x_2^+}{x_2^-}\right)^2 \frac{1}{s(1,2)} \left(\frac{x_1^+}{x_1^-}\frac{x_2^-}{x_2^+}\right)^2 \frac{1}{\sigma(1,2)} = \frac{1}{S_0(1,2)^2}$$

- Scalar factor is unitary
- Unitarity follows from the crossing relations
- The dressing phase is not real due to the AFS phase
- All higher-loop corrections, e.g. the HL one, are real
- Does the BES phase transform properly under the complex conjugation?
- We can split the scalar factor in a product of two unitary factors

$$S_0(1,2)^2 = \frac{x_1^- - x_2^+}{x_1^+ - x_2^-} \frac{1 - \frac{1}{x_1^- x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}} \cdot \left(\frac{1 - \frac{1}{x_1^+ x_2^-}}{1 - \frac{1}{x_1^- x_2^+}}\right)^2 e^{i\theta_{12}}$$

Double-Wick rotation and the generalized rapidity

The S-matrix is written in terms of $x^\pm$ [Beisert,Dippel,Staudacher '04]; [Beisert '04]

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i}{g}$$

This constraint defines a torus Janik '06

The dispersion relation $\mathcal{E}^2 - 4g^2 \sin^2 \frac{p}{2} = 1$

can be solved in terms of elliptic functions

$$p = 2\,\mathrm{am}\;z\,, \quad \sin\frac{p}{2} = \mathrm{sn}(z,k)\,, \quad \mathcal{E} = \mathrm{dn}(z,k)$$

$k = -4g^2 = -\frac{\lambda}{\pi^2} < 0$ is the elliptic modulus (as in *Mathematica*)

The energy is positive and momentum is real because

$$1 \leq \mathrm{dn}(z,k) \leq \sqrt{k'}\,, \quad k' \equiv 1-k \qquad \text{for real } z$$

The torus has real period $2\omega_1$ and imaginary period $2\omega_2$

$$2\omega_1 = 4\mathrm{K}(k)\,, \qquad 2\omega_2 = 4i\mathrm{K}(1-k) - 4\mathrm{K}(k)$$

Double-Wick rotation and the generalized rapidity

The parameters $x^{\pm}$ are expressed in terms of z as follows

$$x^{\pm} = \frac{\mathrm{cn}(z,k) \pm i\,\mathrm{sn}(z,k)}{\sqrt{-k}\ \mathrm{sn}(z,k)}(1 + \mathrm{dn}(z,k))$$

The mirror momentum ($\mathcal{E} \to i\widetilde{p}$) in terms of z

$$\widetilde{p} = -i\,\mathrm{dn}(z,k)$$

For which values of z is $\widetilde{p}$ real?

$$\widetilde{p} = -i\,\mathrm{dn}\left(z + \frac{\omega_2}{2}, k\right) \equiv \sqrt{k'}\,\frac{\mathrm{sn}\,z}{\mathrm{cn}\,z}$$

- For real z the corresponding values of $\widetilde{p}$ are real as well
- The double-Wick rotation is the shift by $2\omega_2/4$
- No periodicity in $\widetilde{p}$ because $\mathrm{cn}(z,k)$ has zeroes at $z = \pm\frac{1}{2}\omega_1$
- The kinematic region in z is from $-\omega_1/2$ to $\omega_1/2$

Double-Wick rotation and the generalized rapidity

$x^{\pm}$ in terms of the shifted parameter z of the mirror model

$$x^{\pm} = -i\frac{\sqrt{k'} \mp \mathrm{dn}(z,k)}{\sqrt{-k}\,\mathrm{dn}(z,k)}\left(1 + i\sqrt{k'}\frac{\mathrm{sn}(z,k)}{\mathrm{cn}(z,k)}\right)$$

$x^{\pm}$ in terms of the mirror momentum

$$x^{\pm} = \frac{1}{\sqrt{-k}}\left(\sqrt{1 - \frac{k}{1+\widetilde{p}^2}} \mp 1\right)(\widetilde{p} - i)$$

Note that for $\widetilde{p}$ real

$$x^+x^- = \frac{\widetilde{p} - i}{\widetilde{p} + i} \in S^1\,, \qquad \frac{x^+}{x^-} \in \mathbf{R}$$

The dispersion relation in the mirror theory takes the form

$$\widetilde{\mathcal{E}} = 2\,\mathrm{arccoth}\frac{\sqrt{k'}}{\mathrm{dn}(z)} = 2\,\mathrm{arcsinh}\frac{1}{\sqrt{-k}}\sqrt{1+\widetilde{p}^2}$$

Double-Wick rotation and the generalized rapidity

Parameter u similar to the rapidity parameter of the XXX spin chain

$$u = x^+ + \frac{1}{x^+} - \frac{i}{g} = x^- + \frac{1}{x^-} + \frac{i}{g}$$

Rapidity u in terms of the shifted parameter z of the mirror model

$$u = -\frac{2i\sqrt{k'}\,\mathrm{dn}\left(z + \frac{\omega_2}{2}, k\right)}{\sqrt{-k}\,\mathrm{dn}\left(z, k\right)}$$

It is real for real z, and in this case we can express it in terms of $\widetilde{p}$

$$u = \frac{2\widetilde{p}}{\sqrt{-k}}\sqrt{1 - \frac{k}{1+\widetilde{p}^2}} = \frac{\widetilde{p}}{g}\sqrt{1 + \frac{4g^2}{1+\widetilde{p}^2}}$$

Physical rectangle is mapped onto the u-plane, and it is one-to-one. Points $z = \omega_1/2 \pm i\,0\,,\ z = -\omega_1/2 \pm i\,0$ are mapped to $u = \pm\infty \ \pm\ i\,\infty$.

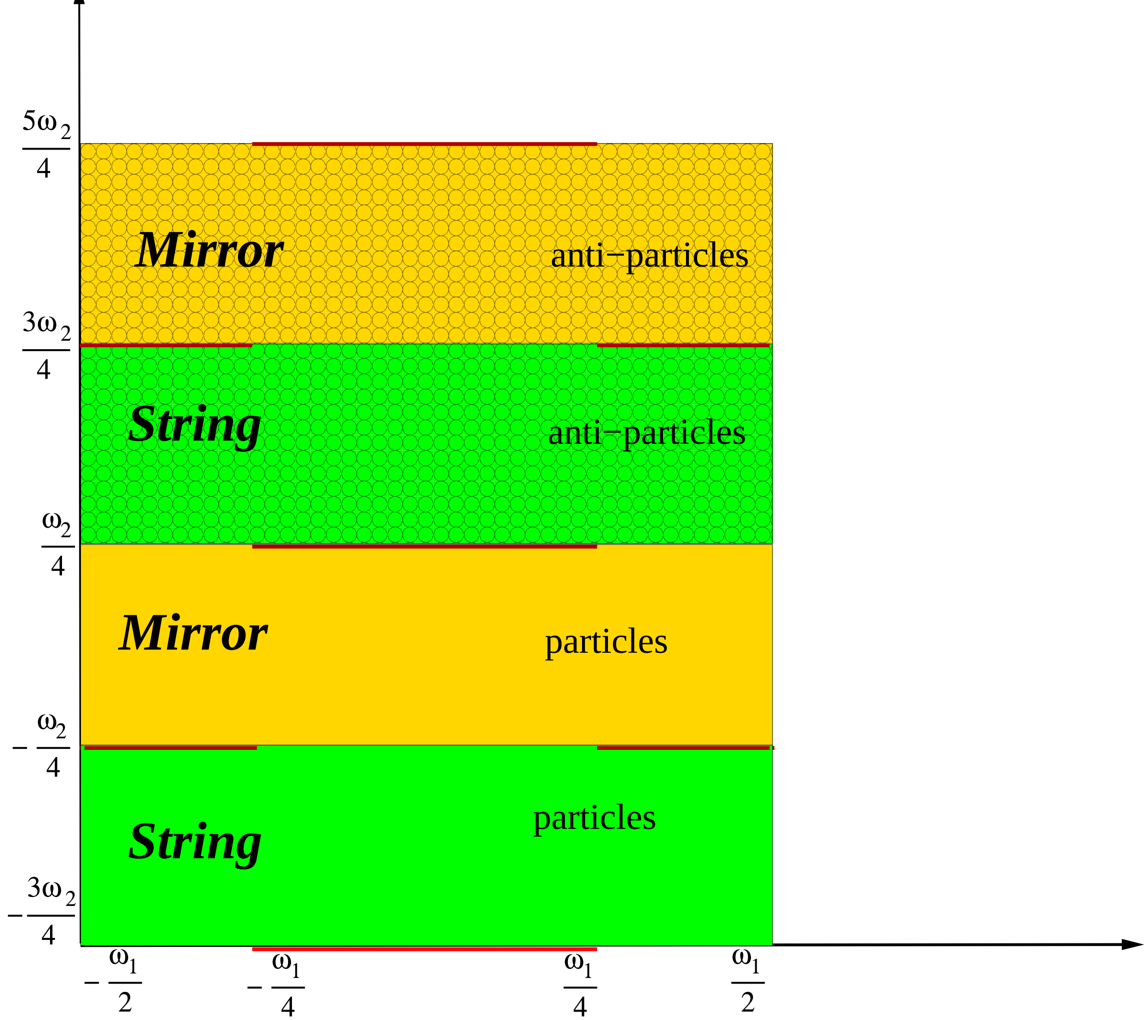

Figure 1: The plot of the torus on which kinematic variable z lives. The (half-)torus $-\frac{1}{2}\omega_1 \leq \text{Re}(z) \leq \frac{1}{2}\omega_1$ and $-\frac{3}{4}\omega_2 \leq \text{Im}(z) \leq \frac{5}{4}\omega_2$ is divided into 4 rectangles. The physically allowed region of strings is the lower green rectangle. Next is the physical rectangle of the mirror theory obtained by a shift by the quarter of the period in the imaginary direction of the torus.

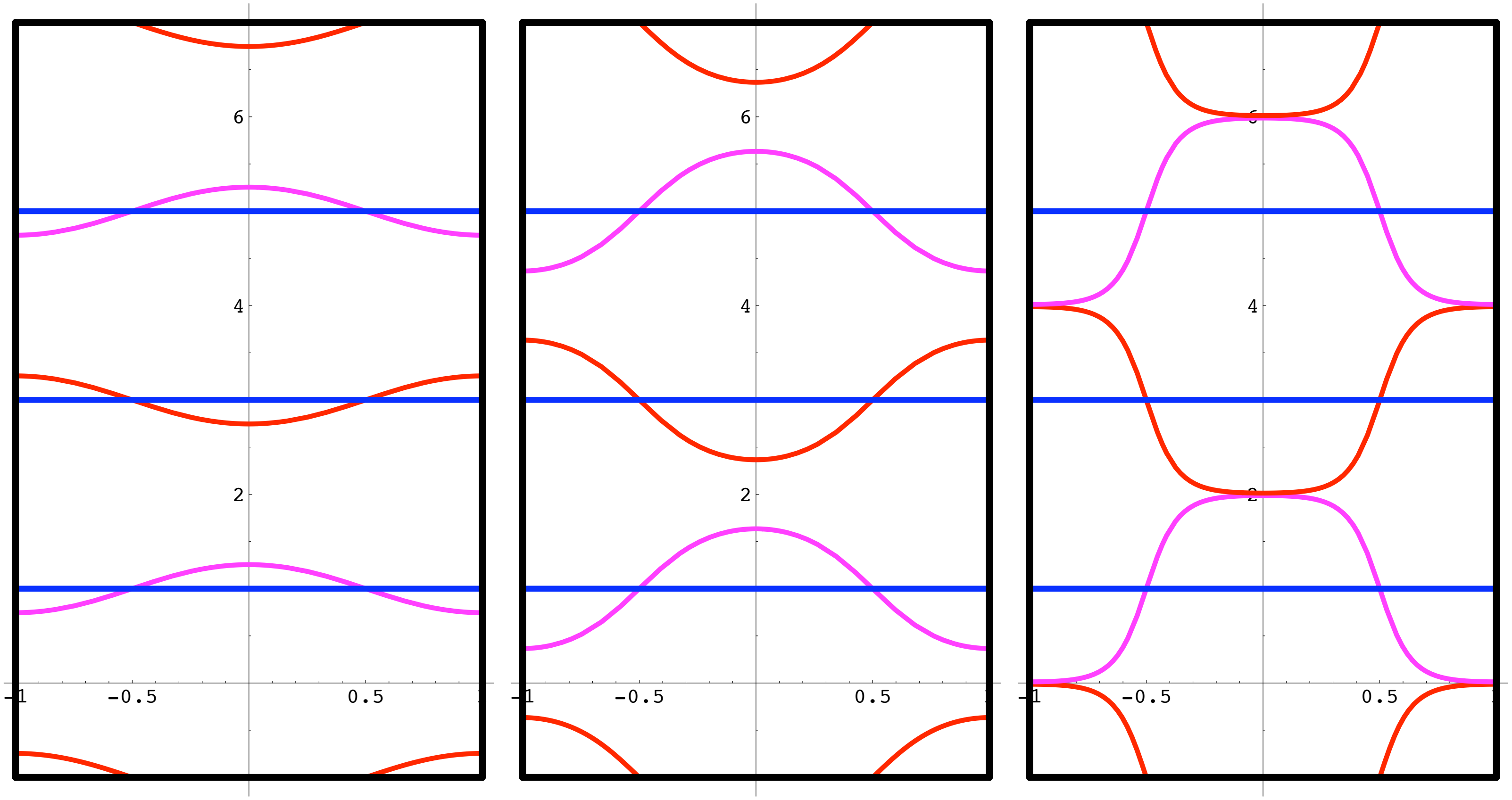

Figure 2: The plot of the curves $|x^{\pm}| = 1$ for $\frac{\lambda}{\pi^2} = 1$, $\frac{\lambda}{\pi^2} = 10$ and $\frac{\lambda}{\pi^2} = 10000$. The red curves are $|x^-| = 1$, and the pink ones are $|x^+| = 1$. The coordinates x and y are rescaled real and imaginary parts of z: $x = Re(\frac{2}{\omega_1} z)$, $y = Re(\frac{4}{\omega_2} z)$.

Bethe Ansatz Equations

- BAE for the mirror system: $S(\widetilde{p}_1, \widetilde{p}_2)$ and $\exp(i\,\widetilde{p}\,R)$
- The form of BAE depends on the choice of the first-level vacuum
- K^I operators $A^{\dagger}_{1\dot{1}}$ acting on $|0\rangle$ create states from $\mathfrak{su}(2)$ sector
- Mirror theory: K^I operators $A^{\dagger}_{3\dot{3}}$ create states from $\mathfrak{sl}(2)$ sector
  - In the original theory this corresponds to $\eta = -1$ Beisert,Staudacher '05
  - $\mathfrak{sl}(2)$ sector is the sector where the mirror magnons exist
  - there are M-particle bound states made only out of the $A^{\dagger}_{3\dot{3}}$-type particles

Bethe Ansatz Equations

[Beisert,Staudacher '05]; [Beisert '05] [Martins,Melo '07]; [de Leeuw '07]

$$e^{i\widetilde{p}_k R} = \prod_{\substack{l=1\\l\neq k}}^{K^{\rm I}} \sigma_{kl} \frac{x_k^- - x_l^+}{x_k^+ - x_l^-} \frac{1 - \frac{1}{x_k^+ x_l^-}}{1 - \frac{1}{x_k^- x_l^+}} \prod_{\alpha=1}^{2} \prod_{l=1}^{K^{\rm II}_{(\alpha)}} \frac{x_k^+ - y_l^{(\alpha)}}{x_k^- - y_l^{(\alpha)}} \sqrt{\frac{x_k^-}{x_k^+}}$$

$$1 = \prod_{l=1}^{K^{\rm I}} \frac{y_k^{(\alpha)} - x_l^+}{y_k^{(\alpha)} - x_l^-} \sqrt{\frac{x_l^-}{x_l^+}} \prod_{l=1}^{K^{\rm III}_{(\alpha)}} \frac{v_k^{(\alpha)} - w_l^{(\alpha)} + \frac{i}{g}}{v_k^{(\alpha)} - w_l^{(\alpha)} - \frac{i}{g}}$$

$$1 = \prod_{l=1}^{K^{\rm II}_{(\alpha)}} \frac{w_k^{(\alpha)} - v_l^{(\alpha)} - \frac{i}{g}}{w_k^{(\alpha)} - v_l^{(\alpha)} + \frac{i}{g}} \prod_{\substack{l=1\\l\neq k}}^{K^{\rm III}_{(\alpha)}} \frac{w_k^{(\alpha)} - w_l^{(\alpha)} + \frac{2i}{g}}{w_k^{(\alpha)} - w_l^{(\alpha)} - \frac{2i}{g}}$$

where ($A^\dagger_{M\dot{N}} \sim A^\dagger_{M,1} \times A^\dagger_{N,2}$)

$$K^{\rm I} = N(A^\dagger_{1,\alpha}) + N(A^\dagger_{2,\alpha}) + N(A^\dagger_{3,\alpha}) + N(A^\dagger_{4,\alpha}) \quad \textit{for any } \alpha$$

$$K^{\rm II}_{(\alpha)} = 2N(A^\dagger_{4,\alpha}) + N(A^\dagger_{1,\alpha}) + N(A^\dagger_{2,\alpha})$$

$$K^{\rm III}_{(\alpha)} = N(A^\dagger_{2,\alpha}) + N(A^\dagger_{4,\alpha})$$

Bound states: strings

- Complex values of momenta; poles in the S-matrix
- In the thermodynamic limit they correspond to Bethe strings

Consider Bethe strings in the $\mathfrak{su}(2)$ sector of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ theory.

Bound state of two particles of type $A^{\dagger}_{1\dot{1}}$ with momenta

$$p_1 = \frac{p}{2} - iq\,, \quad p_2 = \frac{p}{2} + iq\,, \quad q > 0$$

The first BA equation takes the form

$$e^{ipL/2}e^{qL} = e^{iP}\prod_{l=2}^{K^{\mathrm{I}}}\sigma_{1l}\,\frac{x_1^+ - x_l^-}{x_1^- - x_l^+}\,\frac{1-\frac{1}{x_1^+x_l^-}}{1-\frac{1}{x_1^-x_l^+}}\prod_{\alpha=1}^{2}\prod_{l=1}^{K^{\mathrm{II}}_{(\alpha)}}\frac{x_1^- - y_l^{(\alpha)}}{x_1^+ - y_l^{(\alpha)}}$$

where $L = J + K^{\mathrm{I}} - \frac{1}{2}K^{\mathrm{II}}_{(1)} - \frac{1}{2}K^{\mathrm{II}}_{(2)}$ and $P = p_1 + p_2 + \cdots + p_{K^{\mathrm{I}}}$.

Bound states: strings

$$e^{ipL/2}e^{qL} = e^{iP}\prod_{l=2}^{K^{\mathrm{I}}}\sigma_{1l}\,\frac{x_1^+ - x_l^-}{x_1^- - x_l^+}\,\frac{1-\frac{1}{x_1^+x_l^-}}{1-\frac{1}{x_1^-x_l^+}}\,\prod_{\alpha=1}^{2}\prod_{l=1}^{K^{\mathrm{II}}_{(\alpha)}}\frac{x_1^- - y_l^{(\alpha)}}{x_1^+ - y_l^{(\alpha)}}$$

For large L the l.h.s. is exponentially divergent. Assume that x_1^+ is not close to $y_l^{(\alpha)}$ for any l and α. Then, there exists a root p_2 such that

$$(x_1^- - x_2^+)\left(1-\frac{1}{x_1^-x_2^+}\right) \sim e^{-qL} \quad \Rightarrow \quad u_1 - u_2 - \frac{2i}{g} \sim e^{-qL}$$

The exponentially small term is determined by the other Bethe roots.

The reality condition gives the pattern of a Bethe string

$$u_{1,2} = u_0 \pm \frac{i}{g}\,, \qquad u_0 \in \mathbb{R}$$

Bound state condition is satisfied if

$$x_1^- - x_2^+ = 0 \quad or \quad 1-\frac{1}{x_1^-x_2^+} = 0$$

Bound states: strings

$$x_1^- - x_2^+ = 0 \quad or \quad 1 - \frac{1}{x_1^- x_2^+} = 0$$

The reality condition for $x^\pm$ is $\left(x_1^-\right)^* = x_2^+ \quad \Rightarrow$

$$\boxed{Im(x_1^-) = 0} \quad or \quad \boxed{|x_1^-| = 1}$$

Eq. $Im(x_1^-) = 0$ has solutions in the physical rectangle only if

$$|u_0| \geq 2\,, \quad u_{\rm critical} = \pm 2 + \frac{i}{g} = \pm 2 + \frac{2\pi i}{\sqrt{\lambda}}$$

In terms of momentum solutions with real q exist only if

$$\sin^2\frac{p}{2} \leq \frac{1}{2g^2}\left(\sqrt{1+4g^2}-1\right)\,, \quad 0 \leq q \leq \log\frac{2g+\sqrt{2\sqrt{1+4g^2}-2}}{\sqrt{1+4g^2}-1}$$

Energy of the bound state is $E \leq \sqrt{2\sqrt{1+4g^2}+2}$

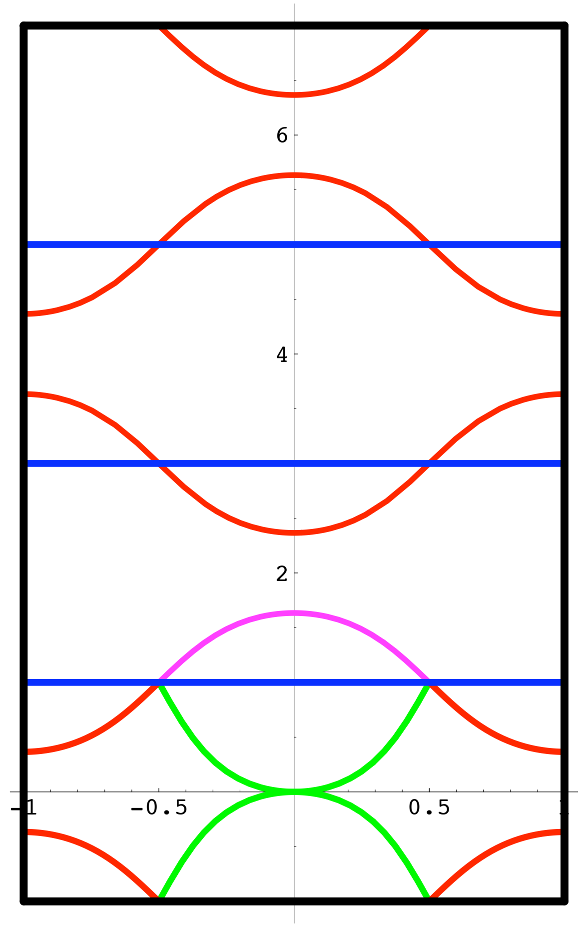

Figure 3: In the physical rectangle of strings the red curves are $|x^{\pm}| = 1$, and the green curves are $Im(x^-) = 0$ for $Im(z) < 0$ and $Im(x^+) = 0$ for $Im(z) > 0$, and $\frac{\lambda}{\pi^2} = 10$. The points of the curves symmetric about the x-axis correspond to z_1 and z_2 of 2-particle bound states.

Bound states: strings

$$|u_0| \geq 2\,, \quad u_{\text{critical}} = \pm 2 + \frac{i}{g} = \pm 2 + \frac{2\pi i}{\sqrt{\lambda}}$$

$$\sin^2\frac{p}{2} \leq \frac{1}{2g^2}\left(\sqrt{1+4g^2}-1\right)\,, \quad E \leq \sqrt{2\sqrt{1+4g^2}+2}$$

In the region $|u_0| \geq 2$ or $p \leq p_{\text{critical}}$, the state is BPS Chen, N.Dorey, Okamura '06

$$E = \sqrt{2^2 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}\,,$$

and belongs to a short irrep satisfying the shortening condition Beisert '06

What about the second region, $|u_0| \leq 2$ or $p \geq p_{\text{critical}}$ corresponding to $|x_1^-| = 1$?

Bound states: strings

If $|u_0| \leq 2$ or $|x_1^-| = 1$ or $p \geq p_{\mathrm{critical}}$

- The bound state energy E does not satisfy the shortening condition
- The state belongs to a long multiplet

Bound states: strings

If $|u_0| \leq 2$ or $|x_1^-| = 1$ or $p \geq p_{\rm critical}$

- The bound state energy E does not satisfy the shortening condition
- The state belongs to a long multiplet
- All 256 states of the multiplet are bound states
- In particular, there are bound states in $\mathfrak{sl}(2)$ and $\mathfrak{su}(1|1)$ sectors

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Bound states: strings

If $|u_0| \leq 2$ or $|x_1^-| = 1$ or $p \geq p_{\text{critical}}$

- The bound state energy E does not satisfy the shortening condition
- The state belongs to a long multiplet
- All 256 states of the multiplet are bound states
- In particular, there are bound states in $\mathfrak{sl}(2)$ and $\mathfrak{su}(1|1)$ sectors
- E as a function of p is not given by any simple explicit formula
- There is an explicit formula of E as a function of u_0

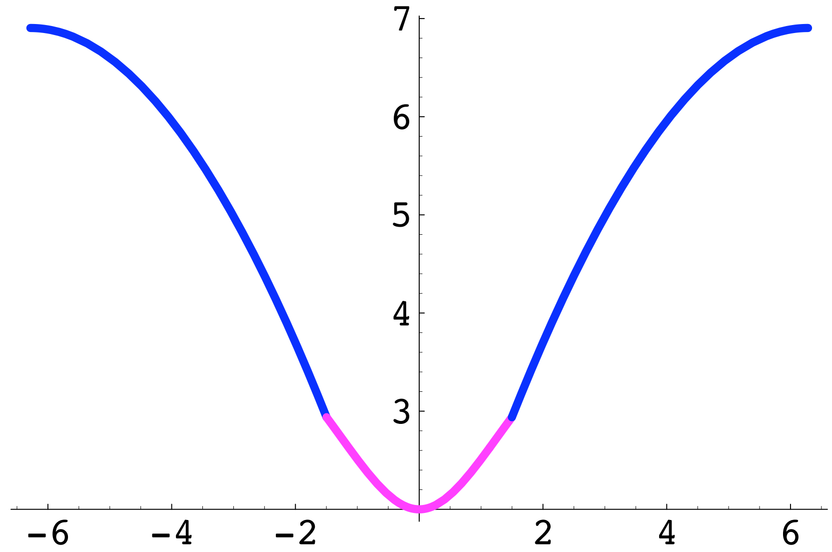

Figure 4: Plot of the energy of a 2-particle bound state as a function of the total momentum for $\frac{\lambda}{\pi^2} = 10$.

- For $p = 2\pi$ we get

$$E = 2(g + \sqrt{1 + g^2}) > 2\sqrt{1 + 4g^2} \approx 4g = 2\frac{\sqrt{\lambda}}{\pi}$$

- If the BPS formula were correct we'd get $E = \frac{\sqrt{\lambda}}{\pi}$

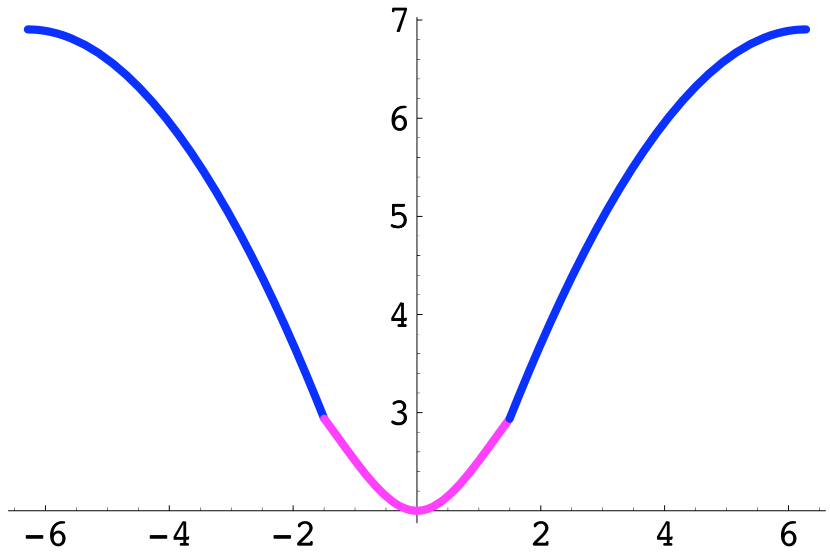

Figure 5: Plot of the energy of a 2-particle bound state as a function of the total momentum for $\frac{\lambda}{\pi^2} = 10$.

- For $p = 2\pi$ we get

$$E = 2(g + \sqrt{1+g^2}) > 2\sqrt{1+4g^2} \approx 4g = 2\frac{\sqrt{\lambda}}{\pi}$$

- If the BPS formula were correct we'd get $E = \frac{\sqrt{\lambda}}{\pi}$
- Is the bound state the breather? Hofman, Maldacena '06
- Our results differ from [N.Dorey '06]; [N.Dorey, Hofman, Maldacena '07]

Bound states: mirror strings

Bound state of two mirror particles of type $A^{\dagger}_{3\dot{3}}$ with momenta

$$\widetilde{p}_1 = \frac{p}{2} - iq\,, \quad \widetilde{p}_2 = \frac{p}{2} + iq\,, \quad q > 0$$

The first BA equation takes the form

$$e^{ipR/2} e^{qR} = \prod_{l=2}^{K^{\rm I}} \sigma_{1l}\, \frac{x_1^- - x_l^+}{x_1^+ - x_l^-}\, \frac{1 - \frac{1}{x_1^+ x_l^-}}{1 - \frac{1}{x_1^- x_l^+}} \prod_{\alpha=1}^{2} \prod_{l=1}^{K^{\rm II}_{(\alpha)}} \frac{x_1^+ - y_l^{(\alpha)}}{x_1^- - y_l^{(\alpha)}} \sqrt{\frac{x_1^-}{x_1^+}}$$

For large R the l.h.s. is exponentially divergent. Assume that x_1^- is not close to $y_l^{(\alpha)}$ for any l and α. Then, there exists a root $\widetilde{p}_2$ such that

$$\left(x_1^+ - x_2^-\right)\left(1 - \frac{1}{x_1^- x_2^+}\right) \sim e^{-qR}$$

The reality condition for $x^{\pm}$ in the mirror theory is $\left(x_1^{\pm}\right)^* = 1/x_2^{\mp} \quad \Rightarrow$

$$\boxed{|x_1^+| = 1} \quad or \quad \boxed{Im(x_1^-) = 0}$$

Bound states: mirror strings

$$\boxed{|x_1^+| = 1} \quad or \quad \boxed{Im(x_1^-) = 0}$$

Eq. $Im(x_1^-) = 0$ has NO solutions in the physical rectangle of mirror theory

Eq. $|x_1^+| = 1$ gives a Bethe string

$$u_{1,2} = u_0 \mp \frac{i}{g}\,, \qquad |u_0| \leq 2$$

In terms of momentum the solutions with real q exist only if

$$p \leq \sqrt{2}\sqrt{-1 + \sqrt{1 + \frac{\lambda}{\pi^2}}}\,, \quad q \leq \frac{1}{\sqrt{2}}\sqrt{1 + \sqrt{1 + \frac{\lambda}{\pi^2}}}$$

The energy is

$$\widetilde{\mathcal{E}} \leq 2\,\mathrm{arcsinh}\frac{\pi}{\sqrt{\lambda}}\sqrt{2^2 + p^2} = 2\,\mathrm{arcsinh}\frac{\pi}{\sqrt{\lambda}}\sqrt{2 + 2\sqrt{1 + \frac{\lambda}{\pi^2}}}$$

Bound states: mirror strings

$$u_{1,2} = u_0 \mp \frac{i}{g}\,, \qquad |u_0| \leq 2$$

We get a restriction on u_0 and the momentum ...?

If we want to have no restriction on u_0, we should have

$$(x_1^+ - x_2^-)\left(1 - \frac{1}{x_1^+ x_2^-}\right) = u_1 - u_2 + \frac{i}{g} = 0$$

Let's take a look at the BA equation again

$$e^{ipR/2} e^{qR} = \prod_{l=2}^{K^{\mathrm{I}}} \sigma_{1l} \, \frac{x_1^- - x_l^+}{x_1^+ - x_l^-} \, \frac{1 - \frac{1}{x_1^+ x_l^-}}{1 - \frac{1}{x_1^- x_l^+}} \prod_{\alpha=1}^{2} \prod_{l=1}^{K^{\mathrm{II}}_{(\alpha)}} \frac{x_1^+ - y_l^{(\alpha)}}{x_1^- - y_l^{(\alpha)}} \sqrt{\frac{x_1^-}{x_1^+}}$$

For large R the l.h.s. is exponentially divergent. Assume that x_1^- is not close to $y_l^{(\alpha)}$ for any l and α. Then, there exists a root $\widetilde{p}_2$ such that

$$(x_1^+ - x_2^-)\left(1 - \frac{1}{x_1^- x_2^+}\right) = 0$$

Bound states: mirror strings

$$u_{1,2} = u_0 \mp \frac{i}{g}\,, \qquad |u_0| \le 2$$

We get a restriction on u_0 and the momentum ...?

If we want to have no restriction on u_0, we should have

$$(x_1^+ - x_2^-)\left(1 - \frac{1}{x_1^+ x_2^-}\right) = u_1 - u_2 + \frac{i}{g} = 0$$

Let's take a look at the BA equation again

$$e^{ipR/2} e^{qR} = \prod_{l=2}^{K^{\mathrm{I}}} \sigma_{1l}\, \frac{x_1^- - x_l^+}{x_1^+ - x_l^-}\, \frac{1 - \frac{1}{x_1^+ x_l^-}}{1 - \frac{1}{x_1^- x_l^+}} \prod_{\alpha=1}^{2} \prod_{l=1}^{K^{\mathrm{II}}_{(\alpha)}} \frac{x_1^+ - y_l^{(\alpha)}}{x_1^- - y_l^{(\alpha)}} \sqrt{\frac{x_1^-}{x_1^+}}$$

There exists a root $\widetilde{p}_2$ such that

$$\frac{1}{\sigma_{12}}\, \frac{x_1^+ - x_2^-}{x_1^- - x_2^+}\, \frac{1 - \frac{1}{x_1^- x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}} = 0$$

Bound states: mirror strings

$$(x_1^+ - x_2^-)\left(1 - \frac{1}{x_1^+ x_2^-}\right) = 0 \quad and \quad \frac{1}{\sigma_{12}} \frac{x_1^+ - x_2^-}{x_1^- - x_2^+} \frac{1 - \frac{1}{x_1^- x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}} = 0$$

Thus σ_{12} should be of the form

$$\sigma_{12} = \widetilde{\sigma}_{12} \left(\frac{1 - \frac{1}{x_1^- x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}} \right)^2$$

where $\widetilde{\sigma}_{12}$ has no zeroes or poles on $(x_1^+ - x_2^-)\left(1 - \frac{1}{x_1^+ x_2^-}\right) = 0$ and no poles in the physical rectangle of the mirror theory, and

$$S_0(1,2)^2 = \frac{x_1^- - x_2^+}{x_1^+ - x_2^-} \frac{1 - \frac{1}{x_1^- x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}} \cdot \widetilde{\sigma}_{12} = \frac{u_1 - u_2 - \frac{i}{g}}{u_1 - u_2 + \frac{i}{g}} \cdot \widetilde{\sigma}_{12}$$

Note that $\widetilde{\sigma}_{12}$ is unitary. We believe it should have a nice expansion in powers of $\frac{1}{x^-}$ and x^+.

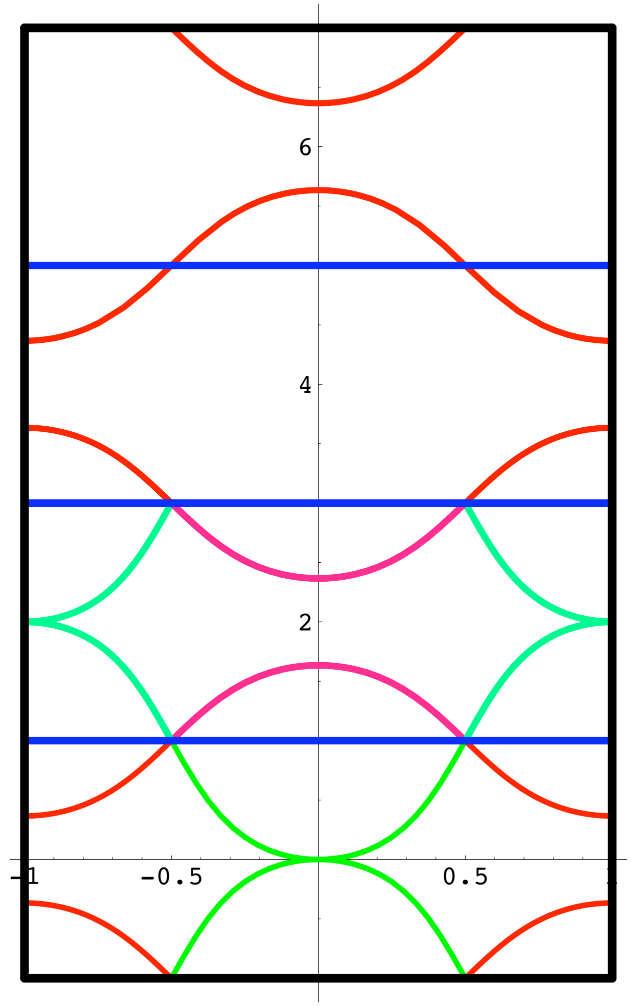

Figure 6: In the physical rectangle of mirror strings the red curves are $|x^{\pm}| = 1$, and the green curves are $Im(x^{\pm}) = 0$, and $\frac{\lambda}{\pi^2} = 10$. The points of the curves symmetric about the horizontal line through the centre of the rectangle correspond to z_1 and z_2 of 2-particle bound states.

Conclusion and Open Problems

- The $AdS_5 \times S^5$ string sigma-model can be naturally embedded in the general framework of massive integrable systems
- Our construction of the $\mathfrak{su}(2|2)$-invariant S-matrix is the conventional field-theoretic one
- The superstring S-matrix shares most of the usual properties of S-matrices arising in relativistic two-dimensional integrable models
- Analyze the structure of the BAE in the thermodynamic limit and write down a system of TBA equations

What will be left of spin chains?