Magnon kinematics in planar $\mathcal{N}=4$ *Yang-Mills*†

Rafael Hernández, IFT-Madrid

† Collaboration with N. Beisert, C. Gómez and E. López

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Introduction

The AdS/CFT correspondence:
The large N limit of $\mathcal{N} = 4$ Yang-Mills is dual to type IIB string theory on $AdS_5 \times S^5$ $\Rightarrow$ Spectra of both theories should agree

$\rightarrow$ Difficult to test, because the correspondence is a strong/weak coupling duality: we can not use perturbation theory on both sides

String energies expanded at large λ

$$E(\lambda) = \lambda^{1/4} E_0 + \lambda^{-1/4} E_1 + \lambda^{-3/4} E_2 + \ldots$$

Scaling dimensions of gauge operators at small λ

$$\Delta(\lambda) = D_0 + \lambda D_1 + \lambda^2 D_2 + \ldots$$

$$\boxed{E(\lambda) \leftrightarrow \Delta(\lambda)}$$

$\rightarrow$ **Integrability:** S_{string} **should interpolate to** S_{gauge}

Integrability in the AdS/CFT correspondence

A complete formulation of the AdS/CFT correspondence $\Rightarrow$ Precise identification of **string states** with local **gauge invariant operators**

$$\Rightarrow E\sqrt{\alpha'} = \Delta$$

Strong evidence in the **supergravity regime**, $R^2 \gg \alpha'$ $(R^4 = 4\pi g_{YM}^2 N \alpha'^2)$

Difficulties:

- String quantization in $AdS_5 \times S^5$
- Obtaining the whole spectrum of $\mathcal{N}=4$ is involved

An insight: [Berenstein, Maldacena, Nastase]

$\Downarrow$

Operators carrying **large charges**, $\boxed{\text{tr}\,(X_1^J \ldots), J \gg 1}$

Verifying AdS/CFT in **large** spin sectors $\Rightarrow$ Computation of the anomalous dimensions of **large operators**

(Difficult problem due to **operator mixing**)

Insightful solution:

$\longrightarrow$ The **one-loop planar dilatation operator** of $\mathcal{N} = 4$ Yang-Mills leads to an integrable spin chain ($SO(6)$ in the scalar sector [Minahan,Zarembo] or $PSU(2,2|4)$ in the complete theory [Beisert,Staudacher])
(Recall Lipatov´s work in QCD)

Single trace operators can be mapped to states in a closed spin chain
$\Rightarrow$ BMN **impurities**: **magnon** excitations

$$\mathrm{tr}(XXXYYX\ldots) \leftrightarrow |\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\ldots\rangle$$

The Bethe ansatz

→ The rapidities u_j parameterizing the momenta of the magnons satisfy a set of **one-loop Bethe equations**

$$e^{ip_j J} \equiv \left(\frac{u_j + i/2}{u_j - i/2}\right)^J = \prod_{k\neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i} \equiv \prod_{k\neq j}^{M} S(u_j, u_k)$$

Thermodynamic limit: **integral equations**

→ **Assuming integrability** an **asymptotic long-range Bethe ansatz** has been proposed [Beisert,Dippel,Staudacher]

$$\left(\frac{x_j^+}{x_j^-}\right)^J = \prod_{k\neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i} = \prod_{k\neq j}^{M} \frac{x_j^+ - x_k^-}{x_j^- - x_k^+} \frac{1 - \lambda/16\pi^2 x_j^+ x_k^-}{1 - \lambda/16\pi^2 x_j^- x_k^+}$$

where $x_j^{\pm}$ are generalized rapidities

$$x_j^{\pm} \equiv x(u_j \pm i/2)\,, \quad x(u) \equiv \frac{u}{2} + \frac{u}{2}\sqrt{1 - 2\frac{\lambda}{8\pi^2}\frac{1}{u^2}}$$

The quantum string Bethe ansatz

String non-linear sigma model on the coset

$$\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}$$

Integrable

[Mandal,Suryanarayana,Wadia] [Bena,Polchinski,Roiban]

$\Rightarrow$ Classical solutions of the sigma model are parameterized by an integral equation [Kazakov,Marshakov,Minahan,Zarembo]

$$-\frac{x}{x^2 - \frac{\lambda}{16\pi^2 J^2}} \frac{\Delta}{J} + 2\pi k = 2\,\mathcal{P} \int_{\mathcal{C}} dx' \frac{\rho(x')}{x - x'} \qquad x \in \mathcal{C}$$

Reminds of the **thermodynamic Bethe equations** for the spin chain ...

The previous string integral equations are
classical/thermodynamic equations

$\Downarrow$

Assuming integrability survives at the quantum level, a discretization would provide a **quantum string Bethe ansatz** [Arutyunov,Frolov,Staudacher]

The **quantum string Bethe ansatz** is [Arutyunov,Frolov,Staudacher]

$$\left(\frac{x_j^+}{x_j^-}\right)^J = \prod_{k\neq j}^{M} \frac{x_j^+ - x_k^-}{x_j^- - x_k^+} \frac{1 - \lambda/16\pi^2 x_j^+ x_k^-}{1 - \lambda/16\pi^2 x_j^- x_k^+} e^{2i\theta(x_j, x_k)}$$

The string and gauge theory ansätze **differ by a dressing phase factor!!**
The phase factor is given by

$$\theta_{12} = 2\sum_{r=2}^{\infty} c_r(\lambda)\Big(q_r(x_1)q_{r+1}(x_2) - q_{r+1}(x_1)q_r(x_2)\Big)$$

($q_r(p_i)$ are the conserved magnon charges

$$q_r(x^{\pm}) = \frac{i}{r-1}\left(\frac{1}{(x^+)^{r-1}} - \frac{1}{(x^-)^{r-1}}\right)$$)

$\longrightarrow$ **The dressing phase coefficients $c_r(\lambda)$ should interpolate from the string to the gauge theory** (strong/weak-interpolation)

$\Downarrow$

Explicit form of $c_r(\lambda)$ ...

To constrain **the string Bethe ansatz** and find the structure of the dressing phase we can compare with **one-loop corrections** to semiclassical strings

$\Downarrow$

The classical limit $c_r(\infty) = 1$ needs to be modified in order to include quantum corrections to the string

Symmetries of the scattering matrix

One-loop corrections to semiclassical strings

→ **One-loop corrections** are obtained from the spectrum of **quadratic fluctuations** around a classical solution [Frolov,Tseytlin] [Frolov,Park,Tseytlin]

→ They amount to empirical constraints on the string Bethe ansatz

→ Careful analysis of the one-loop sums over bosonic and fermionic frequencies [Schäfer-Nameki,Zamaklar,Zarembo] [Beisert,Tseytlin] [RH,López] [Freyhult,Kristjansen]

provides a compact form of the **first quantum correction**

[RH,López] [Gromov,Vieira]

$$\begin{aligned} c_{r,s} &= \delta_{r+1,s} + \frac{1}{\sqrt{\lambda}} a_{r,s} \\ a_{r,s} &= -8\,\frac{(r-1)(s-1)}{(r+s-2)(s-r)} \end{aligned}$$

Crossing symmetry and the dressing phase factor

Crossing symmetry

The structure of the complete S-matrix is [Beisert]

$$\mathcal{S}_{12} = S^0_{12}\left[\, S^{SU(2|2)}_{12}\, S^{SU(2|2)\,'}_{12}\,\right]$$

- Term in the bracket: determined by the **symmetries** (Yang-Baxter)
- The scalar coefficient is the dressing factor: constrained by unitarity and crossing ($\rightarrow$ **dynamics**) [Janik], which implies

$$\theta(x_1^\pm, x_2^\pm) + \theta(1/x_1^\pm, x_2^\pm) = -2i \log h(x_1^\pm, x_2^\pm)$$

with

$$h(x_1, x_2) = \frac{x_2^-}{x_2^+}\,\frac{x_1^- - x_2^+}{x_1^+ - x_2^+}\,\frac{1 - 1/x_1^- x_2^-}{1 - 1/x_1^+ x_2^-}$$

An expansion of both sides has been shown to agree, using the explicit form of the one-loop correction in $\theta(x_1, x_2)$ [Arutyunov,Frolov]

Higher corrections

Idea: Search for coefficients to fit the expansion of the crossing function $h(x_1, x_2)$

This provides a **strong-coupling expansion** [Beisert,RH,López]

$$c_{r,s} = \sum_{n=0}^{\infty} c_{r,s}^{(n)} g^{1-n}$$

for the coefficients in the dressing phase $\left(g \equiv \sqrt{\lambda}/4\pi\right)$

The **all-order** proposal is

$$c_{r,s}^{(n)} = (r-1)(s-1)\,\mathrm{B}_n\,\mathcal{A}(r,s,n)$$

with

$$\mathcal{A}(r,s,n) = \frac{\left((-1)^{r+s}-1\right)}{4\cos(\frac{1}{2}\pi n)\,\Gamma[n+1]\,\Gamma[n-1]} \times \frac{\Gamma[\frac{1}{2}(s+r+n-3)]}{\Gamma[\frac{1}{2}(s+r-n+1)]}\,\frac{\Gamma[\frac{1}{2}(s-r+n-1)]}{\Gamma[\frac{1}{2}(s-r-n+3)]}$$

→ Includes the **classical** and **one-loop** terms

→ An expansion dressed with the Bernoulli numbers

→ The **one-loop contribution** alone satisfies part of the crossing relation (odd crossing)

→ The remaining piece of the crossing condition is satisfied by the ***n*-loop contribution**, with n even

(The **solution** is however **not unique**: it is possible to include additional **homogeneous solutions** to the crossing constraints)

→ The phase shows **agreement with perturbative string theory** (semiclassical scattering of giant magnons [Hofman,Maldacena])

Weak-coupling expansion

→ The previous (strong-coupling) asymptotic expansion

$$c_{r,s} = \sum_{n=0}^{\infty} c_{r,s}^{(n)} g^{1-n}$$

agrees with the string theory regime

→ The **weak-coupling regime** is constrained by perturbative computations:

- Up to three-loops the phase $\theta(x_1, x_2)$ should remain zero
- A recent **four-loop** computation requires a **first non-vanishing piece** in the dressing phase [Bern,Czakon,Dixon,Kosower,Smirnov]

→ The four-loop result can be recovered from a **long-range Bethe ansatz computation** [Beisert,Eden,Staudacher] (See also [Benna,Benvenuti,Klebanov, Sardicchio] [Alday,Arutyunov,Benna,Eden,Klebanov] [Beccaria,DeAngelis,Forini] [Kotikov, Lipatov, Rej, Staudacher, Velizhanin]...)

Quantum–deformed magnon kinematics

The previous succesful interpolation from the strong to the weak-coupling regime relies strongly on the long-range Bethe ansatz of [Beisert,Dippel,Staudacher]

We will now try to address two questions

→ What is the magnon kinematics underlying the long-range ansatz?

→ Is the gauge theory (the correspondence) really integrable?

Clarifying the **features of magnon kinematics** is indeed of great importance

⇓

→ In $1 + 1$ relativistic theories physical conditions are used to constrain the S-matrix: **unitarity, bootstrap principle, crossing symmetry**

→ The remaining traditional condition is **Lorentz covariance** ⇒ Forces dependence on the diference of rapidities

Let us briefly recall the way the long-range Bethe ansatz is constructed

→ The (one-loop) Heisenberg chain has dispersion relation

$$E = 4\sin^2\left(\frac{p}{2}\right)$$

→ The Bethe ansatz can be **deformed** to include the magnon dispersion relation for planar $\mathcal{N} = 4$ Yang-Mills,

$$E^2 = 1 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)$$

The extension/deformation is the long-range Bethe ansatz

[Beisert,Dippel,Staudacher]

We will now try to uncover the **magnon kinematics** underlying planar $\mathcal{N} = 4$ Yang-Mills

→ In a $1+1$-dimensional relativistic theory particles transform in irreps of the Poincaré algebra, $E(1,1)$,

$$[J, P] = E \ , \quad [J, E] = P \ , \quad [E, P] = 0$$

An irrep is specified by a value of the Casimir operator

$$m^2 = E^2 - p^2$$

→ The dispersion relation in planar $\mathcal{N} = 4$ is a deformation of the usual relativistic relation ⇒ **There is an algebra whose Casimir has the adequate form!!**

⇓

It is a quantum deformation of the $1+1$ Poincaré algebra, $E_q(1,1)$

[Gómez,RH] [Young]

$E_q(1,1)$ is the algebra

$$\begin{aligned} KEK^{-1} &= E \;, \quad KJK^{-1} = J + iaE \;, \\ KK^{-1} &= \mathbb{1} \;, \quad JE - EJ = \frac{K - K^{-1}}{2ia} \end{aligned}$$

with **deformation parameter** $q = e^{ia}$ and $K = e^{iaP}$ (the limit $a \to 0$ corresponds to the usual Poincaré algebra)

Furthermore, the boost generator J can be used to introduce a uniformizing rapidity through $J = \frac{\partial}{\partial z}$. Then the algebra implies

$$\frac{\partial p}{\partial z} = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p}{2}\right)}$$

which provides a **elliptic uniformization** in terms of Jacobi functions

[Janik] [Beisert] [RH,Gómez] [Kostov,Serban,Volin]

$$\sin\left(\frac{p}{2}\right) = k' \mathrm{sd}(z) \;, \quad E(z) = \frac{1}{2\mathrm{dn}(z)}$$

Semi–continuum limit of the Ising model

The quantum-deformed Poincaré, or the dispersion relation in planar $\mathcal{N} = 4$ Yang-Mills, can in fact be obtained from the Ising model

→ The lattice spacings a_x and a_t can be mapped to the Ising couplings K and L ($*$ stands for the Kramers-Wannier dual)

$$\sinh 2L \sinh 2K^* = \left(\frac{a_x}{a_t}\right)^2 \quad , \quad 2\sinh(L - K^*) = \mu a_x$$

→ Define $\gamma \equiv p a_x$, $\omega \equiv E a_t$. Then **Onsager´s relation** becomes

$$\cosh\gamma = \cosh 2L \cosh 2K^* - \sinh 2L \sinh 2K^* \cos\omega$$
$$\Downarrow$$
$$a_t^2 \big(\cosh p a_x - 1\big) + a_x^2 \big(\cos E a_t - 1\big) = \tfrac{1}{2}\mu^2 a_t^2 a_x^2$$

→ In the continuum limit $a_x, a_t \to 0$ we get $p^2 + E^2 = \mu^2$

The **semi-continuum limit** $a_t \to 0$ leads to the dispersion relation in planar $\mathcal{N} = 4$ Yang-Mills (after analytical continuation of p and the introduction of an effective scale through a_x)

In fact, the uniformization in planar $\mathcal{N} = 4$ is the same as that in the Ising model (cf [Baxter])

The Boltzmann weights are indeed made out from $x^{\pm}$

$$x^{\pm} = e^{2L} e^{\mp 2K}$$

(map by [Kostov,Serban,Volin])

and **integrability from the star-triangle relation** implies Beisert´s algebraic constraint [Beisert]

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{1}{4ig}$$

Co–multiplication rules

Central Hopf subalgebra

The S-matrix can be determined explicitly

[Beisert]

$\Downarrow$

The spin chain vacuum breaks the $PSU(2,2|4)$ symmetry algebra down to $(PSU(2|2) \times PSU(2|2)') \ltimes \mathbb{R}$, with $\mathbb{R}$ a shared central charge

The $PSU(2|2) \ltimes \mathbb{R}$ algebra is generated by bosonic generators R^a_b and $\mathcal{L}^\alpha_\beta$, and supersymmetry generators Q^α_b and G^a_β

$$\{Q^\alpha_a, G^b_\beta\} = \delta_a{}^b \mathcal{L}^\alpha_\beta + \delta^\alpha_\beta R^b_a + \delta_a{}^b \delta_\beta{}^\alpha c$$

The algebra can be extended with two central charges $\mathfrak{P}$ and $\mathfrak{K}$ (with eigenvalues P and K)

[Beisert]

$$\{Q^{\alpha}_{a}, Q^{\beta}_{b}\} = \epsilon^{\alpha\beta}\epsilon_{ab}\mathfrak{P} , \quad \{G^{a}_{\alpha}, G^{b}_{\beta}\} = \epsilon^{ab}\epsilon_{\alpha\beta}\mathfrak{K} .$$

They define an action on multiple magnon states with a non-trivial co-multiplication (once a new element $\mathfrak{R}$, with eigenvalue e^{ip}, is introduced)
(cf the **dynamic–non local representation** in [Beisert])

[Gómez,RH] [Plefka,Spill,Torrielli]

$$\begin{aligned} \Delta\mathfrak{P} &= \mathfrak{P}\otimes\mathfrak{R} + \mathbb{1}\otimes\mathfrak{P} , \\ \Delta\mathfrak{K} &= \mathfrak{K}\otimes\mathbb{1} + \mathfrak{R}^{-1}\otimes\mathfrak{K} , \\ \Delta\mathfrak{R} &= \mathfrak{R}\otimes\mathfrak{R} \end{aligned}$$

Construction of the S-matrix

In order to construct the S-matrix, **conservation** along a scattering process **of the central charges** is required

[Beisert]

$$\hat{P}_1 + \hat{P}_2 = \hat{P}_1' + \hat{P}_1' \ , \quad \hat{K}_1 + \hat{K}_2 = \hat{K}_1' + \hat{K}_1' \ , \quad \hat{P}_i \hat{K}_i = \hat{P}_i' \hat{K}_i'$$

After bootstrap, the only non-trivial solution is

$$\hat{P}_1 = P_1 \ , \quad \hat{P}_2 = \hat{P}_2 z_1 \ , \quad \hat{P}_2' = P_2 \ , \quad \hat{P}_1' = \hat{P}_1 z_2$$

with $P_i = \alpha(1 - e^{i\, p_i})$, $z_i = e^{ip_i}$ ($\rightarrow$ **non-local**)

$\rightarrow$ **Non-trivial** co-multiplication rule, $\hat{\Delta} = \hat{\mathfrak{P}}_1 + \hat{\mathfrak{P}}_2 = \mathfrak{P}_1 + \mathfrak{P}_2 z_1$

(Identically for $\hat{\mathfrak{K}}_i$)

Now we **label different irreps** through

$$\begin{aligned} \xi_1 &: (\hat{P}_1, \hat{K}_1, \hat{C}_1) \\ \xi_2 &: (\hat{P}_2 z_1, \hat{K}_1 z_1^{-1}, \hat{C}_2) \\ \xi_1' &: (\hat{P}_1 z_2, \hat{K}_1 z_2^{-1}, \hat{C}_1) \\ \xi_2' &: (\hat{P}_2, \hat{K}_2, \hat{C}_2) \end{aligned}$$

and fix the S-matrix by

[Beisert]

$$S\, \Delta_{\xi_1, \xi_2}(a) = \Delta_{\xi_1', \xi_2'}(a)\, S$$

with

$$\Delta_{\xi_1, \xi_2}(a) = \pi_{\xi_1} \otimes \pi_{\xi_2} \Delta(a) \quad \text{and} \quad \Delta(a) = a \otimes \mathbb{1} + \mathbb{1} \otimes a$$

$\rightarrow$ **Different irreps, and trivial co-multiplication rules**

An intertwiner between irreps

Given a Hopf algebra $\mathcal{A}$ with elements a, and **non-trivial** Δ ($\Rightarrow \Delta \neq \Delta'$, with Δ' the permutation), the R-**matrix is the intertwiner**

$$R\Delta(a) = \Delta'(a)R$$

or in terms of irreps

$$R_{\xi_1\xi_2}\pi_{\xi_1} \otimes \pi_{\xi_2}\Delta(a) = \pi_{\xi_2} \otimes \pi_{\xi_1}\Delta(a)R_{\xi_1\xi_2}$$

$$(R_{\xi_1\xi_2} : V_{\xi_1} \otimes V_{\xi_2} \to V_{\xi_2} \otimes V_{\xi_1})$$

We may now wonder how to construct a **Hopf algebra S-matrix** in planar $\mathcal{N} = 4$ Yang-Mills ...

Given non-trivial $\hat{\Delta}(\mathfrak{P})$ and $\hat{\Delta}(\mathfrak{K})$,

→ Lift $\hat{\Delta}(\mathfrak{P})$ and $\hat{\Delta}(\mathfrak{K})$ to the whole algebra

→ Look for S such that

$$S\hat{\Delta}_{\tilde{\xi}_1\tilde{\xi}_2}(a) = \hat{\Delta}_{\tilde{\xi}_2\tilde{\xi}_1}(a)S$$

with **same irreps in |in⟩ and |out⟩ but non-trivial Δ**

We can now compare with

[Beisert]

$$S\Delta_{\xi_1\xi_2}(a) = \Delta_{\xi_1{}'\xi_2{}'}(a)S$$

$$\Rightarrow \text{Equivalent if } \hat{\Delta}_{\tilde{\xi}_1\tilde{\xi}_2}(a) = \Delta_{\xi_1\xi_2}(a)$$

True in the central subalgebra, but not true in general!!

Quantum-deformed Poincaré algebra and strong-coupling expansion

If we define

[Klose,McLoughlin,Roiban,Zarembo]

$$p \equiv \frac{2\pi}{\sqrt{\lambda}} P_{\text{string}}$$

the deformation in quantum Poincaré

[Gómez,RH] [Young]

$$q = e^{ia} = q^{i\,\frac{2\pi}{\sqrt{\lambda}}}$$

and the momentum generator in the central subalgebra described above are given by

$$K = e^{ip} = e^{iaP_{\text{string}}} = q^{P_{\text{string}}}$$

Conclusions

- Testing AdS/CFT in large spin sectors ⇒ Integrability in the planar limit of $\mathcal{N} = 4$ Yang-Mills: Precision tests of the correspondence
- Quantum corrections constrain the string Bethe ansatz
  - Simple form of the first correction
  - A crossing-symmetric phase has been suggested to higher orders
- A proof of the the AdS/CFT correspondence requires identification of spectra, together with interpolation as the coupling evolves
  - The dressing factor interpolates from the string to the gauge theory, and strong to weak-coupling

$$S_{st}(p_j, p_k) = e^{i\theta(p_j,p_k)} S_g(p_j, p_k)$$

- The quantum–deformed plane of magnon kinematics in planar $\mathcal{N} = 4$ Yang-Mills has been identified

Open questions

- **Algebraic origin** of the structure of the dressing phase factor

$$\Downarrow$$

Underlying quantum group symmetry pattern organizing the gauge coupling evolution

[Gómez,RH] [Plefka,Spill,Torrielli] [Arutyunov,Frolov,Plefka,Zamaklar] [Klose,McLoughlin,Roiban,Zarembo] [Torrielli] [Beisert] [Moriyama,Torrielli] ...

- Is the AdS/CFT correspondence really **integrable**?

$$\Downarrow$$

What is the **origin** and **meaning** of integrability in the correspondence?