Exact results for twist operators in planar $\mathcal{N} = 4$ SYM

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Plan of the talk

- Very brief introduction to the Bethe ansatz
- The $\mathfrak{sl}(2)$ subsector
  - General properties
  - Higher charges
  - Exact solutions : twist-two and three
- Non-Linear Integral Equation
  - Motivation and basic ideas
  - One-loop NLIE for $\mathfrak{sl}(2)$ operators
  - Non-Linear Beisert-Eden-Staudacher equation
  - Finite size corrections

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- A convenient choice of the Dynkin diagram is [Beisert, Staudacher '05]

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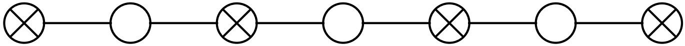

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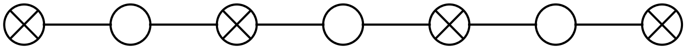

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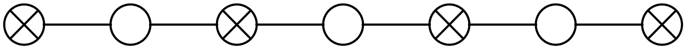

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Asymptotic All-Loop Bethe Equations

$$1 = \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} + \frac{i}{2}}{u_{1,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{1 - g^2/\ x_{1,k} x_{4,j}^+}{1 - g^2/\ x_{1,k} x_{4,j}^-},$$

$$1 = \prod_{\substack{j=1\\j\neq k}}^{K_2} \frac{u_{2,k} - u_{2,j} - i}{u_{2,k} - u_{2,j} + i} \prod_{j=1}^{K_3} \frac{u_{2,k} - u_{3,j} + \frac{i}{2}}{u_{2,k} - u_{3,j} - \frac{i}{2}} \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} + \frac{i}{2}}{u_{2,k} - u_{1,j} - \frac{i}{2}},$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + \frac{i}{2}}{u_{3,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k} - x_{4,j}^+}{x_{3,k} - x_{4,j}^-},$$

$$1 = \left(\frac{x_{4,k}^-}{x_{4,k}^+}\right)^L \prod_{\substack{j=1\\j\neq k}}^{K_4} \left(\frac{u_{4,k} - u_{4,j} + i}{u_{4,k} - u_{4,j} - i}\, \sigma^2(x_{4,k}, x_{4,j})\right)$$

$$\times \prod_{j=1}^{K_1} \frac{1 - g^2/\ x_{4,k}^- x_{1,j}}{1 - g^2/\ x_{4,k}^+ x_{1,j}} \prod_{j=1}^{K_3} \frac{x_{4,k}^- - x_{3,j}}{x_{4,k}^+ - x_{3,j}} \prod_{j=1}^{K_5} \frac{x_{4,k}^- - x_{5,j}}{x_{4,k}^+ - x_{5,j}} \prod_{j=1}^{K_7} \frac{1 - g^2/\ x_{4,k}^- x_{7,j}}{1 - g^2/\ x_{4,k}^+ x_{7,j}},$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k} - u_{6,j} + \frac{i}{2}}{u_{5,k} - u_{6,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{5,k} - x_{4,j}^+}{x_{5,k} - x_{4,j}^-},$$

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- Where [Beisert, Dippel, Staudacher '04]

$$x(u) = \frac{u}{2}\left(1 + \sqrt{1 - \frac{4\,g^2}{u^2}}\right), \qquad x^{\pm} = x(u \pm \frac{i}{2}).$$

- The higher charges are given by

$$Q_r = \frac{i}{r-1}\sum_{j=1}^{K_4}\left(\frac{1}{(x^+(u_j))^{r-1}} - \frac{1}{(x^-(u_j))^{r-1}}\right)$$

- Anomalous dimension corresponds to the Q_2 charge

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The $\mathfrak{sl}(2)$ subsector

- The field content:

$$\mathcal{O} = \text{Tr}\left(\mathcal{D}^M \mathcal{Z}^L\right) + \dots\,,$$

where $\mathcal{D} = \mathcal{D}_1 + i\,\mathcal{D}_2$ and $D_\mu = \partial_\mu + iA_\mu$.

- Asymptotic Bethe equations [Staudacher '04; Beisert, Staudacher '05]

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{M} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-}\,\frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+}\,\sigma^2(u_k, u_j)\,.$$

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- All Bethe roots in this sector are real at weak coupling.
- For a fixed value of L the corresponding anomalous dimension of the ground state has the following asymptotic behavior

$$\gamma(g) = 2\, g^2\, Q_2 = f(g)\, \log M + C(g, L) + \mathcal{O}\left(\frac{1}{M}\right)$$

  at large values of M.
- The scaling function is conjectured to be L independent.
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- It has an interpretation as a fluctuation density [Eden, Staudacher '06]

$$\rho(u) = \rho_0(u) - 8\,g^2\,\frac{\log(M)}{M}\,\sigma(u)$$

- At strong coupling string theory predicts [Gubser, Klebanov, Polyakov '02], [Frolov, Tseytlin, '02]

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Higher charges

- What about the higher charges?
- Starting from the two-loop order they also scale logarithmically

$$Q_r = f_r(g)\,\log(M) + C_r(g,L) + \mathcal{O}\left(\frac{1}{M}\right) \qquad\qquad r = 2,4,6,\ldots$$

- At weak coupling these scaling functions are L independent and also obey the *transcendentality principle*, for example

$$\begin{aligned} f_4(g) &= 16\,\zeta(4)\,g^4 - 16\Big(2\zeta(2)\,\zeta(4) + 15\,\zeta(6)\Big)\,g^6 \\ &+ 32\Big(2\,\zeta(2)^2\,\zeta(4) + 6\,\zeta(4)^2 + 4\,\zeta(3)\,\zeta(5) \\ &+ 15\,\zeta(2)\,\zeta(6) + 98\,\zeta(8)\Big)\,g^8 + \ldots \end{aligned}$$

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Higher charges at strong coupling

- The leading density at strong coupling was found in [Alday, Arutyunov, Benna, Eden, Klebanov '07].

- Integrating it over the charge densities gives

$$f_r(g) = \left(\frac{1}{g}\right)^{r-1} \left(\frac{\Gamma[\frac{r-1}{2}]}{\Gamma[\frac{r}{2}]\,\Gamma[\frac{1}{2}]} - \frac{4}{\pi} \frac{{}_3F_2\left(\frac{3}{2}, \frac{3}{2}-\frac{r}{2}, \frac{1}{2}+\frac{r}{2}; \frac{5}{2}-\frac{r}{2}, \frac{3}{2}+\frac{r}{2}; 1\right)}{(r^2-2r-3)} \right).$$

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Exact solutions

- At one-loop Bethe equations are equivalent to the polynomial solution of the Baxter equation

$$\left(u+\frac{i}{2}\right)^L Q(u+i)+\left(u-\frac{i}{2}\right)^L Q(u-i)=t(u)\,Q(u)\,,$$

where

$$t(u)=2u^L+\sum_{i=2}^{L}\tilde{q}_i\,u^{L-i}\,.$$

- For $L=2$ this equation can be exactly solved [Virginia Dippel, unpublished]

$$Q_2(u)={}_3F_2\left(-M,M+1,\frac{1}{2}+iu,;1,1;1\right)\,.$$

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Twist-three at one-loop

- Surprisingly, one can also solve the Baxter equation for the ground state of $L = 3$ [Kotikov, Lipatov, A.R., Staudacher, Velizhanin, '07], [Beccaria, '07]

$$Q_3(u) = {}_4F_3\Big(-\frac{M}{2}, \frac{M}{2}+1, \frac{1}{2}+iu, \frac{1}{2}-iu, ; 1, 1, 1; 1 \Big) .$$

- Defining the nested harmonic sums:

$$S_a(M) = \sum_{i=1}^{M} \frac{(\text{sgn}(a))^i}{i^{|a|}} , \qquad S_{a_1,\ldots,a_n}(M) = \sum_{i=1}^{M} \frac{(\text{sgn}(a_1))^i}{i^{|a_1|}} S_{a_2,\ldots,a_n}(i) ,$$

- one easily derives the corresponding anomalous dimension

$$\gamma_2(M) = 8\, S_1\left(\frac{M}{2}\right)$$

Twist-three at one-loop

- Surprisingly, one can also solve the Baxter equation for the ground state of $L = 3$ [Kotikov, Lipatov, A.R., Staudacher, Velizhanin, '07], [Beccaria, '07]

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Twist-three at higher loops

- At higher-loops one can derive

$$\frac{\gamma_4^{ABA}(M)}{4} = -2S_3 - 4S_1 S_2\,,$$

$$\frac{\gamma_6^{ABA}(M)}{8} = 2S_2 S_3 + S_5 + 4S_{3,2} + 4S_{4,1} - 8S_{3,1,1} + S_1\left(4S_2^2 + 2S_4 + 8S_{3,1}\right)$$

$$
\begin{aligned}
\frac{\gamma_8^{ABA}(M)}{16} \quad = \quad & S_1^3\left(\frac{40}{3}S_4 - \frac{32}{3}S_{3,1}\right) + S_1^2\left(20S_5 - 40S_{3,2} - 56S_{4,1} + 64S_{3,1,1}\right) \\
& + S_1\left(7S_6 + 8S_{2,4} - 24S_{3,3} - 56S_{4,2} - 40S_{5,1} - 24S_{2,2,2}\right. \\
& -16S_{2,3,1} + 88S_{3,1,2} + 88S_{3,2,1} + 120S_{4,1,1} - 192S_{3,1,1,1} \\
& \left. -8\zeta(3)S_3\right) - \frac{56}{3}S_3S_4 - \frac{107}{6}S_7 + 3S_{2,5} + \frac{41}{3}S_{3,4} + \frac{1}{3}S_{4,3} \\
& -17S_{5,2} - \frac{20}{3}S_{6,1} - 4S_{2,2,3} - 8S_{2,3,2} - 4S_{2,4,1} + \frac{104}{3}S_{3,1,3} \\
& +52S_{3,2,2} + \frac{88}{3}S_{3,3,1} + 60S_{4,1,2} + 60S_{4,2,1} + 40S_{5,1,1} + 8S_{2,3,1,1} \\
& -120S_{3,1,1,2} - 120S_{3,1,2,1} - 120S_{3,2,1,1} - 128S_{4,1,1,1} \\
& +256S_{3,1,1,1,1}
\end{aligned}
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- It is not known how to derive these formulas (and similar for the twist-two case) analytically from the Bethe ansatz.
- There is a detour, however, when one assumes the Kotikov-Lipatov transcendentality principle. [Kotikov, Lipatov '02]
- Curiously, only positive indices of the harmonic sums appear.
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  - A finite M NLBES equation
  - Finite size effects
    - e.g. $\mathcal{O}(M^0)$ corrections
    - A tool for deriving analytically the anomalous dimension in terms of harmonic sums for $L = 2, 3$?
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  - Considering different limits
  - An independent derivation of the BES equation
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- We consider the ground state at fixed L and M.

- The basic step towards constructing the NLIE is to introduce the complementary set of solutions of the $\mathfrak{sl}(2)$ Bethe equations, termed holes.

- The dynamics of the holes is determined by

$$t_g(u_h) = 0\,.$$

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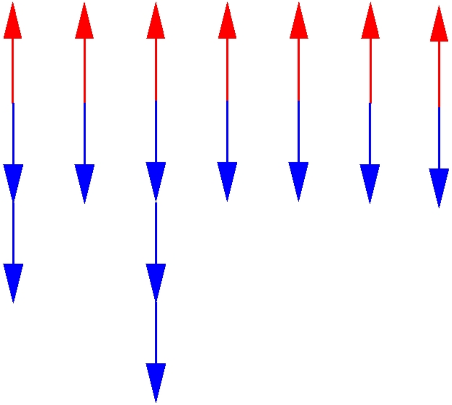
$$\mathrm{Tr}\left(\mathcal{D}^2\mathcal{Z}\ \mathcal{D}\mathcal{Z}\ \mathcal{D}^3\mathcal{Z}\ \mathcal{D}\mathcal{Z}\ \mathcal{D}\mathcal{Z}\ \mathcal{D}\mathcal{Z}\ \mathcal{D}\mathcal{Z}\ \ldots\right)$$



- Two out of these L holes are special.[Belitsky, Korchemsky, Gorsky '06] They scale as

$$u_h^1 = -u_h^2 \simeq \frac{M}{\sqrt{2}}.$$

- They are responsible for the logarithmic scaling of the anomalous dimension and for the universality of the corresponding scaling function.
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One-loop NLIE

- For the one-loop $\mathfrak{sl}(2)$ Bethe equations one defines the counting function as

$$Z(u) = L\phi(u, 1/2) + \sum_{k=1}^{M} \phi(u - u_k) \quad \text{where} \quad \phi(u, \xi) = i \log \left(\frac{i\xi + u}{i\xi - u} \right).$$

- Bethe roots and the holes satisfy

$$e^{iZ(u_k)} = (-1)^{\delta-1} \qquad\qquad k = 1, \ldots, M + L$$

- The following identity is crucial in constructing the NLIE [Feverati, Fioravanti, Grinza, Rossi '06]

$$\sum_{k=1}^{M} f(u_k) \;=\; -\int_{-\infty}^{\infty} \frac{dx}{2\pi} f'(x) Z(x) +$$

$$+ \int_{-\infty}^{\infty} \frac{dx}{\pi} f'(x) \operatorname{Im} \ln \left[1 + (-1)^{\delta} e^{iZ(x+i0)} \right] - \sum_{h=1}^{L} f(x_h).$$

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- Using the above identity for $Z(u)$ one gets the non-linear integral equation

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Z(u) &= iL\log\frac{\Gamma\left(1/2+iu\right)}{\Gamma\left(1/2-iu\right)}+\sum_{j=1}^{L} i\log\frac{\Gamma\left(-i(u-u_h^{(j)})\right)}{\Gamma\left(i(u-u_h^{(j)})\right)} \\
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- At higher loops one defines

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Z(u) &= L i \log \frac{x(i/2+u)}{x(i/2-u)} + i \sum_{k=1}^{M} \log \frac{i+u-u_k}{i-(u-u_k)} \\
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&- \sum_{j=1}^{L}\phi(u-u_h^{(j)},1) - \int_{-\infty}^{\infty}\frac{dv}{\pi}\phi'(u-v,1)\operatorname{Im}\log\left[1+(-1)^{\delta}e^{iZ(v+i0)}\right] \\
&- \int_{-\infty}^{\infty}\frac{dv}{2\pi}\left(-2i\frac{d}{dv}\log\frac{1+\frac{g^2}{x(i/2+u)x(i/2-v)}}{1+\frac{g^2}{x(i/2-u)x_(i/2+v)}}\right)Z(v) \\
&- \sum_{j=1}^{L}\left(-2i\log\frac{1+\frac{g^2}{x(i/2+u)x(i/2-u_h^{(j)})}}{1+\frac{g^2}{x(i/2-u)x(i/2+u_h^{(j)})}}\right) \\
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&+ 2\sum_{r,s}\beta_{r,s}\left(q_r(u)Q_s - q_s(u)Q_r\right).
\end{aligned}
$$

- The conserved charges are related to $Z(u)$ in a similar way, as in the one-loop case

$$Q_p = -\int \frac{dv}{2\pi} q_p'(v) Z(v) - \sum_{j=1}^{L} q_p(u_h^{(j)}) + \int \frac{dv}{\pi} q_p'(v) \operatorname{Im} \log \left[1 + (-1)^{\delta} e^{iZ(v+i0)} \right]$$

and defining

$$\mathbf{Q}_p = g^{p-1}\, i^{p+2} \frac{i}{p-1} \sum_k \left(\frac{1}{(x^+(u_k))^{p-1}} - \frac{(-1)^p}{(x^-(u_k))^{p-1}} \right),$$

one can transform the NLIE equation to

$$
\begin{aligned}
\mathbf{Q}_p &= 2L\int_0^\infty dt\,\frac{J_0(2\,g\,t)J_{p-1}(2\,g\,t)}{t(e^t-1)} \\
&- \sum_{j=1}^{L}\int_0^\infty dt\,\frac{J_{p-1}(2\,g\,t)}{t(1-e^{-t})}\left(e^{-it(u_h^j-i/2)}+e^{it(u_h^j+i/2)}\right) \\
&- 2\int_0^\infty \frac{dt}{\pi}\,\frac{iJ_{p-1}(2\,g\,t)e^{-t/2}}{1-e^{-t}}\hat{L}(t)+\sum_{r=1}^{\infty} r(-1)^{r+1}\mathbf{Q}_{r+1}d_{r+1,p} \\
&- \sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty} 8\Big(2r(2s-1)d_{2r+1,2s}d_{p,2r+1}\mathbf{Q}_{2s} \\
&+ (2r-1)(2s-2)d_{2r,2s-1}d_{p,2s-1}\mathbf{Q}_{2r}\Big).
\end{aligned}
$$

- $d_{r,s}$ are given by

$$
d_{r,s}(g)=\int_0^\infty dt\,\frac{J_{r-1}(2\,g\,t)J_{s-1}(2\,g\,t)}{t(e^t-1)}.
$$

Non-linear BES equation

- In the Fourier space the NLIE generalizes the Beisert-Eden-Staudacher equation to finite values of M

$$
\begin{aligned}
\hat{Z}(t) &= \frac{2\,\pi\,L e^{\frac{t}{2}}}{it(e^t-1)} J_0(2\,g\,t) - \sum_{j=1}^{L} \frac{2\,\pi\,\cos\left(t\,u_h^{(j)}\right)}{it(e^t-1)} - \frac{2}{e^t-1}\,\hat{L}(t) \\
&+ 8\,g^2 \frac{e^{\frac{t}{2}}}{e^t-1} \int_0^\infty dt'\,e^{-\frac{t'}{2}}\,K(2\,g\,t, 2\,g\,t') \Bigg(t'\,\hat{L}(t') \\
&- \frac{\pi}{i} \sum_{j=1}^{L} \cos\left(t'\,u_h^{(j)}\right) \Bigg) \\
&- 4\,g^2 \frac{e^{\frac{t}{2}}}{e^t-1} \int_0^\infty dt'\,e^{-\frac{t'}{2}}\,t'\,K(2\,g\,t, 2\,g\,t')\,\hat{Z}(t')
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- It is a simple application of the above-presented equations to calculate the $\mathcal{O}(M^0)$ to the anomalous dimension at arbitrary loop order

$$
\begin{aligned}
C(g,L) &= \gamma\, f(g) - 8(7-2L)\zeta(3)g^4 + 8\Big(\frac{4-L}{3}\pi^2\zeta(3) \\
&+ (62-21L)\zeta(5)\Big)g^6 - \frac{8}{15}\Big((13-3L)\pi^4\zeta(3) \\
&+ 5(32-11L)\pi^2\zeta(5) + 75(127-46L)\zeta(7)\Big)g^8 \\
&+ 32\left(\frac{4}{945}(49-11L)\pi^6\zeta(3) - (14-4L)\zeta(3)^3\right. \\
&+ \frac{1}{180}(310-103L)\pi^4\zeta(5)\Big) + \Big(\frac{5}{12}(64-5L)\pi^2\zeta(7) \\
&+ \frac{49}{4}(146-55L)\zeta(9)\Big)g^{10} + \dots
\end{aligned}
$$

BES from NLBES

- In the large M limit the NLBES simplifies to

$$\hat{Z}(t') = 8\,\pi\,i\,g^2 \frac{e^{\frac{t}{2}}}{e^t - 1} K(2\,g\,t, 0)\,\log(M) - 4\,g^2 \frac{e^{\frac{t}{2}}}{e^t - 1} \int_0^\infty dt'\,e^{-\frac{t'}{2}}\,t'\; K(2\,g\,t, 2\,g\,t')\,\hat{Z}(t')$$

- Under the identification

$$\hat{Z}(t) = 8\,\pi\,i\,g^2\,e^{\frac{t}{2}}\,\frac{\sigma(t)}{t}\,\log(M)$$

one recovers the BES equation

$$\hat{\sigma}(t) = \frac{t}{e^t - 1}\left(K(2\,g\,t, 0) - 4\,g^2 \int_0^\infty dt'\,K(2\,g\,t, 2\,g\,t')\hat{\sigma}(t')\right)$$

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Conclusions

- All the conserved charges of the $\mathfrak{sl}(2)$ subsector scale as $\log(M)$ at large values of M. This offers the possibility to compare the *whole* integrable structure on both sides of planar AdS/CFT.
- In some special cases the anomalous dimension can be explicitly found as function of M. It would be interesting to investigate what are the precise conditions for such hyperintegrability.
- The Non-Linear Beisert-Eden-Staudacher equation for the ground states of $\mathfrak{sl}(2)$ was derived.
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More conclusions

- With the use of this equation one can derive up to the wrapping order the $\mathcal{O}(M^0)$ corrections to the anomalous dimension of twist operators.
- It offers the possibility to derive independently the Beisert-Eden-Staudacher equation.
- It would be interesting to generalize this method to
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