Perturbative study of the transfer matrix of string theory in $AdS_5 \times S^5$

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Motivations

Goal:	Proving the AdS/CFT correspondence
Slightly more modest goal:	Proving planar AdS/CFT (spectral AdS/CFT)
Immense Progress:	S-matrix of AdS/CFT [AFS][Staudacher][Beisert]
	[Beisert, Hernández, López] [Beisert, Eden, Staudacher]
Shortcomings:	• Integrability?
	S-matrix program assumes factorized scattering
	• S-matrix describes asymptotic spectrum
	$\mathcal{N} = 4$ SYM: [Kotikov, Lipatov, Rej, Staudacher, Velizhanin]
	$AdS_5 \times S^5$: [SSN, Zamaklar, Zarembo]

Showing integrability may seem like a theoretical musing, however describing the exact string/SYM spectra seems to crucially depend on this.

This talk:
Explore quantum integrable structure of string theory in $AdS_5 \times S^5$.

Plan

1. Introduction

- Transfer matrix and non-local charges
- Recap: $O(n)$ model and radiative corrections to the non-local charges
- Why pure spinors are useful

2. Near-flat space limit of $AdS_5 \times S^5$

- Expansion of the currents and action
- OPE of currents

3. Radiative corrections to the transfer matrix

- General sources of log-divergences for Wilson line type operators
- Log-divergences in the transfer matrix of $AdS_5 \times S^5$

4. Summary and Conclusions

1. Introduction

Let J be a flat connection. The transfer matrix is defined as the Wilson line operator

$$\Omega[\Gamma] = P \exp\left(\int_\Gamma J\right)$$

Classically: $J(z)$ = Lax pair, and the monodromy matrix $\Omega(z = 1)^{-1}\Omega(z)$ with closed contour Γ yields local and non-local conserved charges by expanding in the spectral parameter around $z = 0, \infty$ and $z = 1$, respectively.

Question: Does $\Omega[\Gamma]$ remain well-defined (finite) after quantization?

In view of applicability of S-matrix and factorized scattering, only the non-local charges are of interest. Finiteness of $\Omega[\Gamma]$ would imply that the non-local charges survive quantization. (Note, there are examples, when this is not true, $\mathbb{C}P^n$ model).

To discuss the complete spectrum, including finite-size effects one can try applying TBA [Ambjorn, Janik, Kristjansen]. However, this seems to be limited to a finite subclass of states.

In the sinh-Gordon model one can compute the exact finite-size spectrum using the method of the Baxter Q-operator [Bytsko, Teschner]. Key ingredient: transfer matrix, R-matrix of the quantum theory.

Here: explore properties of Ω in the case of $AdS_5 \times S^5$ in view of this approach.

Note, that in the pure spinor formalism, it was argued that the bi-local charge is quantum BRST invariant [Berkovits]. Finiteness will follow from our analysis (perturbatively).

Our point of view is however, to show that the monodromy matrix, which seems to be more fundamental than the non-local charges, is non-renormalized.

Recap: non-local charges in the $O(n)$-model

1. Classical integrability:

- $x(\tau, \sigma) \in S^n$ with the action

$$S = \int d\tau d\sigma \partial_i x^\mu \partial^i x_\mu$$

- Conserved current

$$j_i^{\mu\nu} = x^\mu \partial_i x^\nu - x^\nu \partial_i x^\mu$$

- Classical bi-local charge:

$$Q_2^{\mu\nu} = \int_{-\infty}^{\infty} d\sigma_1 d\sigma_2 \, sgn(\sigma_1 - \sigma_2) j_\tau^{\mu\rho}(\tau, \sigma_1) j_\tau^{\nu\rho}(\tau, \sigma_2) - \int_{-\infty}^{\infty} d\sigma \, j_\sigma^{\mu\nu}(\tau, \sigma)$$

- Higher charges are generated by Poisson brackets of $Q_2^{\mu\nu}$

$\rightarrow$ Classical integrability

2. Quantum integrability:

• $Q_2^{\mu\nu}$ has short-distance singularities in the OPE of two $j^{\mu\nu}$:

$$[j_\tau^{\mu\rho}(\sigma_1), j_\tau^{\nu\rho}(\sigma_2)] \sim \frac{1}{\sigma_1 - \sigma_2} j_\sigma^{\mu\nu}(\sigma_2)$$

Point-splitting $\longrightarrow$ $\log(\epsilon)$-divergence

• cancelled by regularizing with $\log(\epsilon) \int j_\sigma$. [Lüscher]

$$Q_2^{\mu\nu} = \int_{-\infty}^{\infty} d\sigma_1 d\sigma_2 \, sgn(\sigma_1 - \sigma_2) j_\tau^{\mu\rho}(\tau, \sigma_1) j_\tau^{\nu\rho}(\tau, \sigma_2) - \log(\epsilon) \int_{-\infty}^{\infty} d\sigma \, j_\sigma^{\mu\nu}(\tau, \sigma)$$

• Higher charges are generated by Poisson brackets of $Q_2^{\mu\nu}$

• Implication of quantum non-local charges:

$\rightarrow$ Factorized scattering (S-matrix [Zamolodchikov^2])
$\rightarrow$ Absence of particle production.

Non-local charges and transfer matrix for $AdS_5 \times S^5$

Standard setup: The GSMT formalism

GS string with κ-symmetry and light-cone gauge fixed. [Metsaev, Tseytlin]

1. Classical integrability:

• Local and non-local charges constructed
[Bena, Polchinski, Roiban][Beisert, Kazakov, Sakai, Zarembo]

2. Quantum integrability:

- S-matrix seems to work beautifully [Beisert], [BHL], [BES]
- Evidence for factorization [Klose, McLoughlin, Roiban,Zarembo]
- BUT how to generalize Lüscher's analysis: OPE of currents?

Why one might contemplate using pure spinors...

GSMT formalism

- Classically integrable
- Quantization: κ/light-cone gauge

→ breaks global $\mathfrak{psu}(2,2|4)$

→ conformal invariance broken

- Immense progress:

semi-classical methods, S-matrix

- Higher genus: light-cone SFT?

Berkovits pure spinor formalism

- Classically integrable (equivalent)
- Quantization: as 2d CFT

→ unbroken $\mathfrak{psu}(2,2|4)$

→ conformal invariance intact

- Very little results in actual computations
- Higher genus: well-defined theory

→ Flat-space: suprising higher-loop results

→ Could be key to non-planar AdS/CFT

Some caveats:

- Pure spinor string in $AdS_5 \times S^5$ is a highly non-trivial, interacting CFT
- "modified Berkovits formalism" (including b-ghost) has not been worked out for $AdS_5 \times S^5$
- Higher-genus: requires understanding of curved $\beta\gamma$-systems [Berkovits, Nekrasov]

Our approach:

- Perturbative expansion around flat-space in curvature corrections: $1/R$
- Perturbatively determine the OPE of currents
- Compute the short-distance singularities of the transfer matrix and non-local charges

2. Near-flat space limit

- Lightning review of classical pure spinor strings in $AdS_5 \times S^5$

- Near-flat space expansion

- OPE of currents

Pure spinor string

1. Matter

$g \in PSU(2,2|4)$ with $g \equiv hg$ for $h \in SO(4,1) \times SO(5)$.

- Capital-currents $J_\pm = -\partial_\pm g g^{-1}$.
  Invariant under global $PSU(2,2|4)$ right-action.
- $\mathbb{Z}_4$-grading of $\mathfrak{psu}(2,2|4)$ yields $\mathfrak{psu}(2,2|4) = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} \oplus \mathfrak{g}_{\bar{2}} \oplus \mathfrak{g}_{\bar{3}}$
- Generators $\{t^0_{[\mu\nu]}\,,\ t^1_{\dot\alpha}\,,\ t^2_\mu\,,\ t^3_\alpha\,\}$, where $\mu = 0, \cdots, 9$, $\alpha, \dot\alpha = 1, \cdots, 16$
- $a \neq 0$: $J_{a+} \to hJ_{a+}h^{-1}$ under $g \to hg$. Thus the small-case currents $j_{a+} = g^{-1}J_{a+}g$ are $SO(4,1) \times SO(5)$-gauge invariant.

2. Ghosts

Bosonic ghosts (λ_3, w_{1+}) and (λ_1, w_{3-}) taking values in $\mathfrak{g}_{\bar{1}} \oplus \mathfrak{g}_{\bar{3}}$.

- Ghost currents: $N_{0+} = -\{w_{1+}, \lambda_3\}$ and $N_{0-} = -\{w_{3-}, \lambda_1\}$ in $\mathfrak{g}_{\bar{0}}$
- Pure spinor constraint: $\Gamma^\mu_{\alpha\beta}$= $SO(9,1)$ gamma-matrices

$$\lambda_1 \Gamma^\mu \lambda_1 = \lambda_3 \Gamma^\mu \lambda_3 = 0$$

Pure spinor string action:
[Berkovits, Bershadsky, Hauer, Zhukov, Zwiebach][Berkovits]

$$S = \frac{R^2}{\pi} \int d^2z \, \mathrm{Str}_{4|4} \Big(\frac{1}{2} J_{2+} J_{2-} + \frac{3}{4} J_{1+} J_{3-} + \frac{1}{4} J_{3+} J_{1-}$$
$$+ w_{1+} \partial_- \lambda_3 + w_{3-} \partial_+ \lambda_1 + N_{0+} J_{0-} + N_{0-} J_{0+} - N_{0+} N_{0-} \Big)$$

Physical spectrum = $H_{Q_{BRST}}$ where classically

$$Q_{BRST} = \int \mathrm{Str} \left(\lambda_1 J_{3-} d\bar{z} + \lambda_3 J_{1+} dz \right)$$

Features:

- Quantum BRST-invariance [Berkovits]
- Quantum conformal invariance [Berkovits][Valillo]

Zero-curvature formulation of the classical theory

Lax pair:

$$J_+(z) = J_{0+} - N_{0+} + \frac{1}{z} J_{3+} + \frac{1}{z^2} J_{2+} + \frac{1}{z^3} J_{1+} + \frac{1}{z^4} N_{0+}$$

$$J_-(z) = J_{0-} - N_{0-} + zJ_{1-} + z^2 J_{2-} + z^3 J_{3-} + z^4 N_{0-}$$

Classical equations are equivalent to

$$[\partial_+ + J_+(z)\,,\ \partial_- + J_-(z)] = 0$$

Note: even after setting $\lambda = w = 0$ this is different from the Lax pair in [Bena, Polchinski, Roiban]: $J_+^{BPR}(z) = J_{0+} + zJ_{1+} + \frac{1}{z} J_{3+} + \frac{1}{z^2} J_{2+}$

Nevertheless, the classical theories are equivalent (by choosing the gauge $J_{1+} = J_{3-} = 0$)

Pure spinor string has interesting combination of conformal and integrable structure!

Near-flat space expansion

Choice of Viel-bein

$$g = \exp\left(\frac{1}{R}\vartheta_L^{\alpha} t_{\alpha}^{3} + \frac{1}{R}\vartheta_R^{\dot{\alpha}} t_{\dot{\alpha}}^{1}\right) \exp\left(\frac{1}{R} x^{\mu} t_{\mu}^{2}\right) = \exp\left(\frac{1}{R}\vartheta_L + \frac{1}{R}\vartheta_R\right) \exp\left(\frac{1}{R} x\right)$$

In the flat-space limit

- boosts and rotations: $t_{[\mu\nu]}^{0}$
- translations: t_{μ}^{2}
- left- and right-moving supersymmetries: t_{α}^{3} and $t_{\dot{\alpha}}^{1}$

Expansion of the Capital-currents for large R:

$$
\begin{aligned}
-J_{2+} &= \frac{1}{R}\partial_+ x + \frac{1}{2R^2}[\vartheta_L, \partial_+\vartheta_L] + \frac{1}{2R^2}[\vartheta_R, \partial_+\vartheta_R] + +O\left(\frac{1}{R^3}\right) \\
-J_{3+} &= \frac{1}{R}\partial_+\vartheta_L + \frac{1}{R^2}[\vartheta_R, \partial_+ x] + O\left(\frac{1}{R^3}\right) \\
-J_{1+} &= \frac{1}{R}\partial_+\vartheta_R + \frac{1}{R^2}[\vartheta_L, \partial_+ x] + O\left(\frac{1}{R^3}\right) \\
-J_{0+} &= \frac{1}{2R^2}[x, \partial_+ x] + \frac{1}{2R^2}[\vartheta_R, \partial_+\vartheta_L] + \frac{1}{2R^2}[\vartheta_L, \partial_+\vartheta_R] + O\left(\frac{1}{R^3}\right).
\end{aligned}
$$

Yields the matter part of the action:

$$
\begin{aligned}
S &= \frac{1}{\pi}\int d^2v \left(\frac{1}{2}C_{\mu\nu}\partial_+ x^\mu \partial_- x^\nu + C_{\dot\alpha\beta}\partial_+\vartheta_R^{\dot\alpha}\partial_-\vartheta_L^{\beta}\right. \\
&\quad \left. -\frac{1}{2}\frac{1}{R} f_{\mu\alpha\beta}\partial_+ x^\mu \vartheta_L^\alpha \partial_-\vartheta_L^\beta - \frac{1}{2}\frac{1}{R} f_{\mu\dot\alpha\dot\beta}\partial_- x^\mu \vartheta_R^{\dot\alpha}\partial_+\vartheta_R^{\dot\beta} + O\left(\frac{1}{R^2}\right)\right)
\end{aligned}
$$

where $C_{AB} = \mathrm{Str}(t_A t_B)$.

And we need the next order as well...

OPEs

The leading order yields the OPE of the elementary fields x, ϑ_L and ϑ_R

$$\begin{aligned} \langle x^\mu(w,\bar{w}) x^\nu(0)\rangle &= -C^{\mu\nu}\log|w|^2 \\ \langle \vartheta_R^{\dot\alpha}(w,\bar{w})\vartheta_L^{\beta}(0)\rangle &= -C^{\dot\alpha\beta}\log|w|^2 \end{aligned}$$

To compute the OPE of the currents $J_\pm$ we need to evaluate

- OPE of the elementary fields
- contributions from the interactions in the action ($\rightarrow$ mixing of left and right movers)

Note: Some matter OPEs were also determined independently using background field method in [Puletti]

The OPE of the currents J_+ are e.g.:

$$
\begin{aligned}
J_{1+}^{\dot\alpha}(w_1)J_{2+}^{\mu}(w_2) &= \frac{1}{R^3}\frac{\partial_+\vartheta_L^{\gamma}}{w_1-w_2}f_{\gamma}^{\ \ \dot\alpha\mu}+\frac{1}{2R^3}\frac{\bar w_1-\bar w_2}{(w_1-w_2)^2}\partial_-\vartheta_L^{\gamma}f_{\gamma}^{\ \ \dot\alpha\mu}+O\left(\frac{1}{R^4}\right)\\
J_{3+}^{\alpha}(w_3)J_{2+}^{\mu}(w_2) &= \frac{2}{R^3}\frac{\partial_+\eta_R^{\dot\beta}}{w_3-w_2}f_{\dot\beta}^{\ \ \alpha\mu}+O\left(\frac{1}{R^4}\right)\\
J_{1+}^{\dot\alpha}(w_a)J_{1+}^{\dot\beta}(w_b) &= -\frac{1}{R^3}\frac{\partial_+x^{\mu}}{w_a-w_b}f_{\mu}^{\ \ \dot\alpha\dot\beta}+O\left(\frac{1}{R^4}\right)\\
J_{3+}^{\alpha}(w_a)J_{3+}^{\beta}(w_b) &= -\frac{2}{R^3}\frac{\partial_+x^{\mu}}{w_a-w_b}f_{\mu}^{\ \ \alpha\beta}-\frac{1}{R^3}\frac{\bar w_a-\bar w_b}{(w_a-w_b)^2}\partial_-x^{\mu}f_{\mu}^{\ \ \alpha\beta}+O\left(\frac{1}{R^4}\right)\\
J_{1+}^{\dot\alpha}(w_1)J_{3+}^{\alpha}(w_3) &= -\frac{1}{R^2}\frac{1}{(w_1-w_3)^2}C^{\dot\alpha\alpha}+O\left(\frac{1}{R^4}\right)\\
J_{2+}^{\mu}(w_m)J_{2+}^{\nu}(w_n) &= -\frac{1}{R^2}\frac{1}{(w_m-w_n)^2}C^{\mu\nu}+O\left(\frac{1}{R^4}\right)\\
J_{0+}^{[\mu\nu]}(w_0)J_{1+}^{\dot\alpha}(w_1) &= -\frac{1}{2R^3}\left(\frac{\vartheta_R^{\dot\beta}(w_0)}{(w_0-w_1)^2}+\frac{\partial_+\vartheta_R^{\dot\beta}(w_0)}{(w_0-w_1)}\right)f_{\dot\beta}^{\ \ \dot\alpha[\mu\nu]}+O\left(\frac{1}{R^4}\right)\\
J_{0+}^{[\mu\nu]}(w_0)J_{3+}^{\alpha}(w_3) &= -\frac{1}{2R^3}\left(\frac{\vartheta_L^{\beta}(w_0)}{(w_0-w_3)^2}+\frac{\partial_+\vartheta_L^{\beta}(w_0)}{(w_0-w_3)}\right)f_{\beta}^{\ \ \alpha[\mu\nu]}+O\left(\frac{1}{R^4}\right)\\
J_{0+}^{[\mu\nu]}(w_0)J_{2+}^{\lambda}(w_2) &= -\frac{1}{2R^3}\left(\frac{x^{\kappa}(w_0)}{(w_0-w_2)^2}+\frac{\partial_+x^{\kappa}(w_0)}{(w_0-w_2)}\right)f_{\kappa}^{\ \ \lambda[\mu\nu]}+O\left(\frac{1}{R^4}\right)
\end{aligned}
$$

Field renormalization:

We find that x has a non-trivial renormalization proportional to $\partial_+ x \partial_- x$ coming from the cubic and quartic interaction vertices:

$$
\begin{aligned}
&-\frac{1}{2}([\partial_+ x, \vartheta_L], \partial_- \vartheta_L) - \frac{1}{2}([\partial_- x, \vartheta_R], \partial_+ \vartheta_R) + \frac{1}{6}(\partial_+ x, [x, [x, \partial_- x]]) \\
&+\frac{1}{2}(\partial_+ x, [\vartheta_L, [\vartheta_R, \partial_- x]]) + \frac{1}{2}(\partial_+ x, [\vartheta_R, [\vartheta_L, \partial_- x]]) \\
&-\frac{1}{4}(\partial_+ x, [\vartheta_R, [\vartheta_L, \partial_- x]]) - \frac{3}{4}(\partial_+ x, [\vartheta_L, [\vartheta_R, \partial_- x]])
\end{aligned}
$$

Log-divergence:

$$
-\frac{1}{6}\frac{1}{R^2}\log \epsilon^2 \, (\partial_+ x, [t^2_\mu, [t^2_\mu, \partial_- x]]) - \frac{1}{2}\frac{1}{R^2}\log \epsilon^2 \, C^{\alpha\dot\beta}(\partial_+ x, \{t^3_\alpha, [t^1_{\dot\beta}, \partial_- x]\})
$$

Replace x by x^{ren}:

$$
x = x^{ren} + \frac{1}{6}\frac{1}{R^2}\log \epsilon^2 \, [t^2_\mu, [t^2_\mu, x^{ren}]] + \frac{1}{2}\frac{1}{R^2}\log \epsilon^2 \, C^{\alpha\dot\beta}\{t^3_\alpha, [t^1_{\dot\beta}, x^{ren}]\}
$$

Ghosts

Ghost action:

$$S_{ghosts} = \frac{1}{\pi} \int d^2v \, \mathrm{Str}\, (w_{1+}(\partial_- \lambda_3 + [J_{0-}, \lambda_3]) + w_{3-}(\partial_+ \lambda_1 + [J_{0+}, \lambda_1]) - N_+ N_-)$$

with ghost currents:

$$N_+^{[\mu\nu]} = -\frac{1}{R^2} \{w_+, \lambda\} = -\frac{1}{R^2} w_+^{\dot{\alpha}} \lambda^\beta f_{\dot{\alpha}\beta}{}^{[\mu\nu]}$$

OPE of pure spinor ghosts:

$$\langle \lambda^\alpha(w, \bar{w}) w_+^{\dot{\beta}}(0) \rangle = \frac{C^{\alpha\dot{\beta}}}{w}$$

OPE of ghost currents:

$$N_+^{[\mu_1\nu_1]}(v) N_+^{[\mu_2\nu_2]}(0) = \frac{1}{R^2} \frac{1}{v} f_{[\mu_3\nu_3]}{}^{[\mu_1\nu_1][\mu_2\nu_2]} N_+^{[\mu_3\nu_3]} + \frac{1}{R^4} \frac{c}{v^2} + \dots ,$$

Renormalization of Wilson-line type operators

The transfer matrix is similar to Wilson-line operators on the world-sheet. So we will study the possible divergence for this type of operator and then apply it to the transfer matrix of $AdS_5 \times S^5$. Consider

$$\Omega[\Gamma] = P\exp\left(-\int_\Gamma J\right)$$

Let $J = O(1/R)$ and expand in powers of $1/R$: this yields and infinite series of multiple ordered integrals $\int \cdots \int J(\tau_1) \cdots J(\tau_n)$.

Divergences that can occur in expectation values of such an operator: standard renorm-group lore

- $\frac{1}{\epsilon}$: depend on the regularization scheme
- $\log(\epsilon)$: independent of regularization scheme

Consider now a classically conformally invariant Wilson-line type operator. In the quantum theory, this operator gets regularized, with counter term C_ϵ

$$\Omega^{reg}[\Gamma] = \lim_{\epsilon\to 0}\Omega_\epsilon[\Gamma] + C_\epsilon[\Gamma]$$

Let $\Gamma' = \lambda\Gamma$. If conformal invariance is maintained in the quantum theory, we have

$$\Omega^{reg}[\Gamma] = \Omega^{reg}[\Gamma']$$

Furthermore $\Omega_\epsilon[\Gamma] = \Omega_{\lambda\epsilon}[\Gamma']$. Thus, in order for Ω^{reg} to be conformally invariant, we have to have

$$C_\epsilon[\Gamma] = C_{\lambda\epsilon}[\Gamma']$$

This is true, for linear divergences $\frac{\int d\tau^+}{\epsilon}$ is invariant under dilatations. But not for log-divergences!

Thus: in order to maintain conformal invariance, all log-divergences have to cancel.

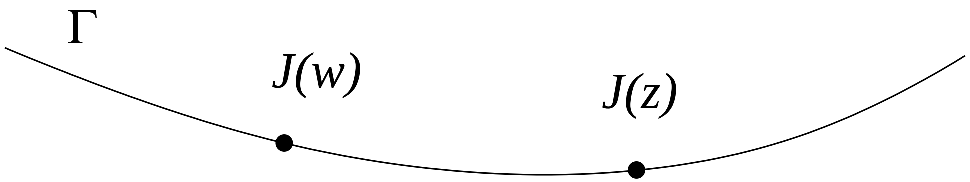
Collision types

Assume OPE of the form

$$j_+^a(w)\, j_+^b(0) \sim -\frac{1}{R^2} C^{ab} \frac{1}{w^2} + F_c^{ab} \frac{1}{w} k_+^c(0)$$

with $[t^a, t^b] = f^{ab}{}_c t^c$ and $F_c^{ab} = -F_c^{ba}$.

- Double collisions:



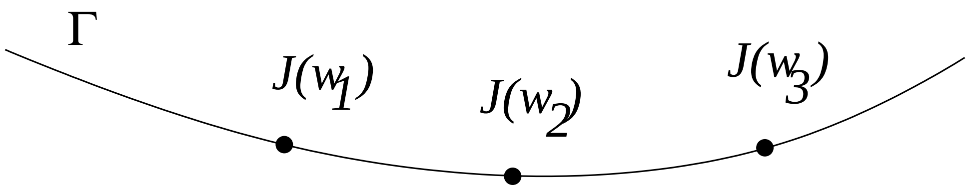
$$\int_{-\infty}^{\infty} dw\ j_+^a(w) t_a \int_{-\infty}^{w-\epsilon} dz\ j_+^b(z) t_b + \int_{-\infty}^{\infty} dw\ j_+^b(w) t_b \int_{-\infty}^{w-\epsilon} dz\ j_+^a(z) t_a$$

$$\longrightarrow -\frac{1}{2} \ln \epsilon \int_{-\infty}^{\infty} dw\ k_+^a(w) F_a^{bc} f_{bc}^e t_e$$

$\frac{1}{w}$-term yields log-divergence. Contributes to non-local charge Q_2.
This is the same situation as in [Lüscher]

•Triple collisions:

Only second order pole of OPE contributes.



Consider first one contraction of two currents in Ω:

$$\Omega = P\left[\left(-\frac{1}{2R^2}\int dw_1 \int dw_2 \frac{C^{ab}t^a(w_1)t^b(w_2)}{(w_1-w_2)^2}\right) : \exp\left(\int j_+ d\tau^+\right) :\right]$$

$t^a(w_1)$= current associated to t^a is inserted at position w_1 along Γ.

$$j_+(a_i) \longrightarrow t^a \longleftrightarrow t^b \longrightarrow j_+(a_{i+1}) : \qquad \int_{a_i+2\epsilon}^{a_{i+1}-\epsilon} dw_2 \int_{a_i+\epsilon}^{w_2-\epsilon} dw_1 \frac{1}{(w_1-w_2)^2} \qquad = +\log\epsilon$$

$$j_+(a_i) \longrightarrow t^a \longrightarrow j_+(a_{i+1}) \longrightarrow t^b \longleftrightarrow j_+(a_{i+2}) : \qquad \int_{a_i+\epsilon}^{a_{i+1}-\epsilon} dw_1 \int_{a_{i+1}+\epsilon}^{a_{i+2}-\epsilon} dw_2 \frac{1}{(w_1-w_2)^2} \qquad = -\log\epsilon$$

$$\longrightarrow -\frac{1}{2}\frac{1}{R^2}\log\epsilon\left(P\left[\int dw\, C^{ab}[t^a,[t^b,j_+]]\exp\left(\int j_+ dw\right)\right]\right)$$

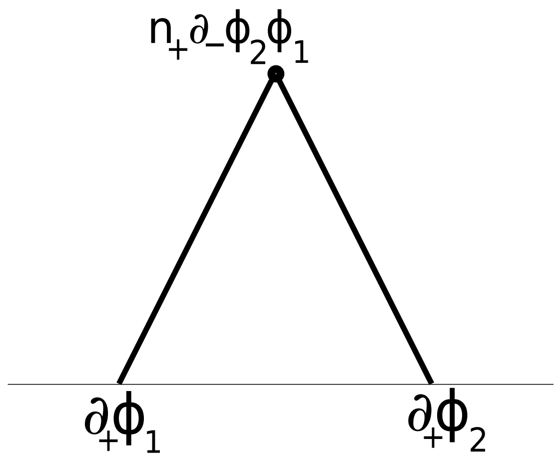
• Triple collisions with field-dependent $1/w^2$

• Divergences from the interaction terms

Consider the following toy model:

$$\frac{1}{\pi}\int d\tau_+ d\tau_- \left(\partial_+\phi_1\partial_-\phi_2 + \frac{1}{R}n_+(\tau_+,\tau_-)\phi_1\partial_-\phi_2\right)$$

Effect on double collisions:



$(\partial_+\phi, \partial_+\phi) \to n_+ :$ $\quad \frac{1}{R}\log\epsilon\int d\tau_+\, n_+$

Similarly: $(+,-) \to n_+$, and $(-,+) \to n_+$, but also, surprisingly $(-,-) \to n_+$.

Log-divergences in the $AdS_5 \times S^5$ transfer matrix

Stratagem: Recall

$$\Omega(\Gamma) = P\exp\left(\int J(z)\right)$$

where z = spectral parameter. Possible log-divergences can be studied at fixed order in the (finite) z-expansion.

Sanity Check: Global charges are not renormalized

Global charges are determined from the transfer matrix on the infinite line $\lim_{\tau_l\to+\infty,\tau_r\to-\infty}(\Omega_{\tau_r}^{\tau_l}(z=1))^{-1}\,\Omega_{\tau_r}^{\tau_l}(z)$, which can be written in terms of the small-case currents $j = g^{-1}Jg$. The global charges are

$$q_{global} = \int *g_{z=1}^{-1}\left[\frac{\partial}{\partial z}\bigg|_{z=1} J(z)\right] g_{z=1}$$

Indeed: divergences proportional to $\partial_+ x$, $[x, \partial_+ x]$ and N_+ all cancel.

Sample computation of log-divergences

Consider e.g. the coefficient of $\frac{1}{z^6} J_{2+}$:

- Double collisions:
  $\frac{1}{z^3} J_{1+}^{\dot\alpha}(w_a) \frac{1}{z^3} J_{1+}^{\dot\beta}(w_b) = -\frac{1}{R^3} \frac{\partial_+ x^\mu}{w_a - w_b} f_\mu{}^{\dot\alpha\dot\beta} + \cdots$
  $$\longrightarrow \frac{1}{z^6}\frac{1}{2}\frac{1}{R^3} \log\epsilon \partial_+ x^\mu C^{\dot\alpha\alpha}\{t^1_{\dot\alpha}, [t^3_\alpha, t^2_\mu]\} = \frac{1}{z^6}\frac{1}{4}\frac{1}{R^3} \log\epsilon\, (C_{\bar 1} + C_{\bar 3}).\partial_+ x$$
  Note, that the $J_{1+}J_{1+}$-OPE depends crucially on the interaction term $-\frac{1}{2R}\partial_+ x[\vartheta_L \partial_- \vartheta_L]$
- Triple collisions:
  $\left[\frac{1}{z^2} J_{2+} \leftrightarrow \frac{1}{z^2} J_{2+} \leftrightarrow \frac{1}{z^2} J_{2+}\right]$ and $\left[\frac{1}{z^3} J_{1+} \leftrightarrow \frac{1}{z^2} J_{2+} \leftrightarrow \frac{1}{z} J_{3+}\right]$
  $$\longrightarrow -\frac{1}{z^6}\frac{1}{2}\frac{1}{R^3} \log\epsilon\, (C_{\bar 1} + C_{\bar 3} + C_{\bar 2}).\partial_+ x$$

$C_{\bar i}$= Casimirs in $\mathfrak{g}_i$.

Casimir identity $C_{\bar 1} + C_{\bar 3} = -2C_{\bar 2}$ when acting in $\mathfrak{g}_2$.

Thus: total $\log\epsilon$-divergences proportional to $\frac{1}{z^6} J_{2+}$ cancel.

Summary of results

- Some cancellations of log-divergences are due to algebraic identities in $\mathfrak{psu}(2,2|4)$ – in particular, vanishing of adjoint Casimir
- At order $\frac{1}{R^3}$ the log-divergences either cancel (see J_{2+} example) or are total derivatives. The total derivatives can be interpreted as coming from a z-dependent gauge transformation

$$\Omega(z) = f(\epsilon, z)\Omega(z)^{finite} f(\epsilon, z)^{-1}$$

with $f(z, \epsilon) = \exp\left(-\frac{1}{2R^2}\left(z^2 + \frac{1}{z^2}\right)\log\epsilon\, C_{\bar{2}}.x + \ldots\right)$

- At order $\frac{1}{R^4}$ there are potential log-divergences proportional to N_+ and to $[x, \partial x]$, which cancel. Both cancellations are rather non-trivial, and probe the structure of the Lax connection and interaction terms.

All in all, to this order in perturbation into $AdS_5 \times S^5$ the transfer matrix seems to be well-defined on the quantum level, and finite.

Conclusions and Outlook

- Evidence for quantum integrability: in particular, Q_2 finiteness reconfirmed and perturbative finiteness of Ω shown
- Developed methods for renormalization of Wilson line type operators
- What are the commutation relations of the transfer matrix? $RTT = TTR$ relations?

Thank you!