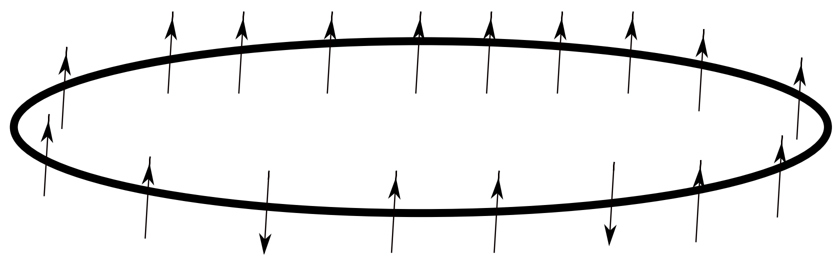
Finite size giant magnon redux

Gordon W. Semenoff

University of British Columbia

D.Astolfi, V.Forini, G.Grignani, G.Semenoff **hep-th/0702043**

Problem of finding the spectrum of planar $\mathcal{N} = 4$ Yang-Mills has a spin-chain analogy
(J.Minahan and K.Zarembo **hep-th/0212208**)



For example: scalar fields of $\mathcal{N} = 4$ super-conformal YM: $\Phi^1, ..., \Phi^6$

$$Z = \Phi^1 + i\Phi^2 \quad , \quad \Phi = \Phi^3 + i\Phi^4 \quad , \quad \Psi = \Phi^5 + i\Phi^6$$

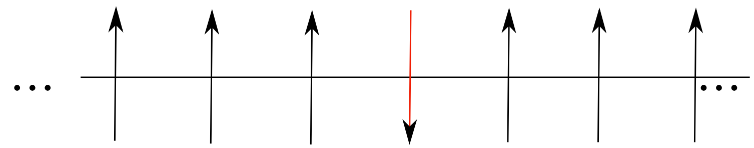
Large N planar limit: conformal dimensions of composite operators

$$\mathrm{Tr}\,[Z(0)Z(0)\Phi(0)Z(0)\Phi(0)Z(0)...] \quad J\ Z's + M\ \Phi's$$

Classical limit $\Delta = J + M$
YM interactions: $\Delta = J + M + \lambda(\text{one loop}) + \lambda^2(\text{two loops}) + \dots$
Resolving degeneracy $\sim$ spin chain

- The spin chain is thought to be integrable and solvable using a Bethe Ansatz
- Problem is simpler in the large volume (magnon) limit
  - planar Yang-Mills theory $N \to \infty$, $\lambda = g_{\rm YM}^2 N$ fixed
  - infinite volume $J \to \infty$ with magnon momenta and λ fixed
- Bethe Ansatz has distinct quasi-particles. In magnon limit integrability implies scattering with a factorized S-matrix.
- quasi-particle is a magnon
- 2-body S-matrix almost completely determined by (super-)symmetry.
- once infinite J spectrum is known – reconstruct finite J
- properties of single magnon

Magnon



$$\sum_x e^{ipx} ...ZZZ\Phi ZZZ...$$

infinitely long spin chain – isolate a single magnon

Spectrum:

$$E = \Delta - J = \sqrt{1 + \frac{\lambda}{\pi} \sin^2 \frac{p_{\rm mag}}{2}}$$

$p_{\rm mag}$ =magnon momentum

Spectrum of magnon:

$$E = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_{\rm mag}}{2}}$$

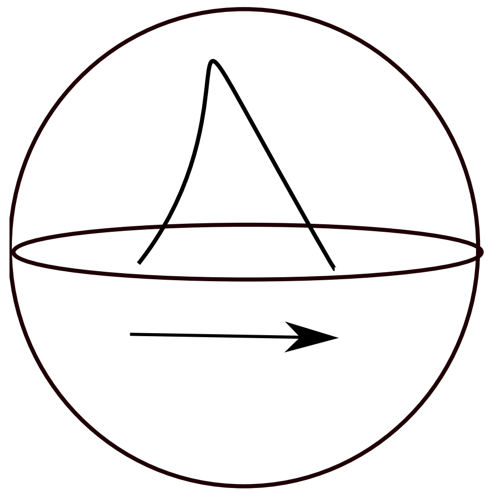
- Compatible with perturbative YM to three loops
- integrability Ansätze at large J
- agrees with BMN limit $p_{\rm mag} = k/J \quad \lambda' = \frac{\lambda}{J^2}$, k, λ' finite.
- Beisert: magnon is $\frac{1}{2}$−BPS state of centrally extended superalgebra $SU(2|2) \times SU(2|2) \times R^3$
- $\lambda \to f(\lambda)$???

Strong coupling limit $\lambda \to \infty$ from string dual $\longrightarrow$

Hofman and Maldacena **hep-th/0604135** identified string dual:

Giant Magnon:

Soliton solution of classical string on $R^1 \times S^2$



angle coordinate open $\phi(r) - \phi(-r) = p_{\text{mag}} \quad \phi'$, all others periodic

$J \;\; (\; = \; -i\partial/\partial\phi) \;\; \to \;\; \infty \quad \theta(\pm r) \to \pi/2$

$E = \frac{\sqrt{\lambda}}{\pi} \left| \sin \frac{p_{\text{mag}}}{2} \right| \quad \leftarrow \quad \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_{\text{mag}}}{2}}$ at large λ

What about corrections to the large J limit?

Finite size corrections?

- Perturbative gauge theory – none! – at least for $J > \#$loops.
- Bethe Ansatz – maybe? – the integrable **Hubbard model** agrees with perturbation theory to a few loops, then is extrapolated to large λ and large J,

$$E_{\mathrm{H}} = \frac{\sqrt{\lambda}}{\pi}\left|\sin\frac{p_{\mathrm{mag}}}{2}\right| \cdot \left[1 - \frac{2\pi^2}{\lambda \sin^2 p_{\mathrm{mag}}/2} e^{-2\pi J/\sqrt{\lambda}|\sin p_{\mathrm{mag}}/2|} + \ldots\right]$$

- finite size and strong coupling from string – apparently yes! Arutyunov, Frolov, Zamlakar **hep-th/0606126**

$$E = \frac{\sqrt{\lambda}}{\pi}\left|\sin\frac{p_{\mathrm{mag}}}{2}\right| \cdot \left[1 - \frac{4}{e^2}\sin^2\frac{p_{\mathrm{mag}}}{2} e^{-\mathcal{R}} + \ldots\right]$$

- Hubbard model matches exponent, $\mathcal{R} = 2\pi J/\sqrt{\lambda}|\sin p_{\mathrm{mag}}/2| + a p_{\mathrm{mag}} \cot p_{\mathrm{mag}}/2$ but not pefactor
- wrapping interactions (e.g. R.Janik, this conference)

- **Finite size corrections are interesting:**
- exponential suppression at large J non-perturbative in Yang-Mills., $\sim e^{-J/\sqrt{\lambda}}$
- **but depend on gauge-fixing parameter a**

  Arutyunov, Frolov, Zamlakar **hep-th/0606126**

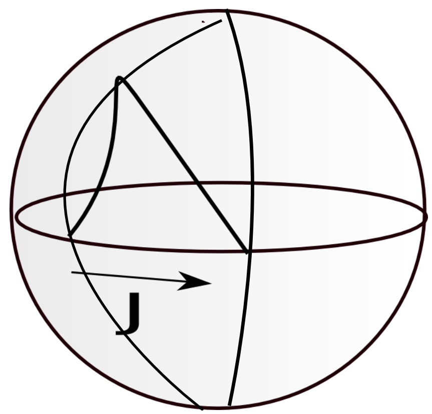
$$\begin{aligned} E &= \frac{\sqrt{\lambda}}{\pi}\left|\sin\frac{p_{\rm mag}}{2}\right| \cdot \Big[1 - \frac{4}{e^2}\sin^2\frac{p_{\rm mag}}{2} e^{-\mathcal{R}} - \\ &- \frac{4}{e^4}\sin^2\frac{p_{\rm mag}}{2}\Big(\mathcal{R}^2(1+\cos p) + 2\mathcal{R}(2+3\cos p_{\rm mag} + \\ &+ a p_{\rm mag}\sin p_{\rm mag}) + 7 + 6\cos p_{\rm mag} + 6 a p_{\rm mag}\sin p_{\rm mag} + \\ &+ a^2 p_{\rm mag}^2 (1-\cos p_{\rm mag})\Big) e^{-2\mathcal{R}} + \ldots\Big] \end{aligned}$$

$$\mathcal{R} = 2\pi J/\sqrt{\lambda}|\sin p_{\rm mag}/2| + a p_{\rm mag}\cot p_{\rm mag}/2$$

There is no state of $N=4$ SYM dual to a giant magnon with $J<\infty$. **Gauge theory dual of finite size giant magnon?**

Orbifold $AdS_5 \times S^5 \;\rightarrow\; AdS_5 \times S^5/Z_M$

Identify longitude on 2-sphere by the action of a discrete group Z_M: $\phi \rightarrow \phi + 2\pi/M$



Non-interacting strings:

- choose subset of momenta $J =$ integer $\cdot\, M$ (rather than $J =$integer in un-orbifold)
- Include wrapped strings $\Delta\phi = 2\pi m/M$

Giant magnon = wrapped closed string

Open ends of magnon are identified identified: $p_{\rm mag} = 2\pi m/M$.
(subset of giant magnon states with $J = kM$)

Orbifold $AdS_5 \times S^5/Z_M$ is dual to $\mathcal{N}=2$ superconformal quiver gauge theory (with $SU(2,2|2)$ superalgebra).

Begin with $\mathcal{N}=4$: Embed regular representation of Z_M into SU(N) gauge group, $N = N'M$

$$\gamma = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & \omega & 0 & ... & 0 \\ 0 & 0 & \omega^2 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & \omega^{M-1} \end{bmatrix} \quad , \quad \omega = \exp(2\pi i/M)$$

(each entry is multiplied by $N' \times N'$ unit matrix)
Keep only those components of fields which are invariant under combined gauge transformation and R-symmetry

$$(Z, \Psi, \Phi, A^\mu) = \left(\omega\gamma Z\gamma^{-1}, \omega^{-1}\gamma\Psi\gamma^{-1}, \gamma\Phi\gamma^{-1}, \gamma A^\mu\gamma^{-1}\right)$$

$$J : Z \to e^{i\theta} Z \quad , \quad J' : \psi \to e^{-i\theta'}\Psi \quad , \quad J + J' = kM$$

For each $N \times N$ matrix field in the parent $\mathcal{N} = 4$ theory,
$M \;\; N' \times N'$-dimensional blocks survive

$$Z = \begin{bmatrix} 0 & Z_1 & 0 & ... & 0 \\ 0 & 0 & Z_2 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ . & . & . & ... & Z_{M-1} \\ Z_M & 0 & 0 & ... & 0 \end{bmatrix}, \; \Phi = \begin{bmatrix} \Phi_1 & 0 & 0 & ... & 0 \\ 0 & \Phi_2 & 0 & ... & 0 \\ 0 & 0 & \Phi_3 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & \Phi_{M-1} \end{bmatrix}$$

$$\Psi = \begin{bmatrix} 0 & 0 & 0 & ... & \Psi_1 \\ \Psi_2 & 0 & 0 & ... & 0 \\ 0 & \Psi_3 & 0 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & ... & \Psi_M & 0 \end{bmatrix}, \; A^\mu = \begin{bmatrix} A_1^\mu & 0 & 0 & ... & 0 \\ 0 & A_2^\mu & 0 & ... & 0 \\ 0 & 0 & A_3^\mu & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & A_{M-1}^\mu \end{bmatrix}$$

Single trace operators from $\mathcal{N} = 4$ notation $\mathrm{Tr}\gamma^m \mathcal{O}$
planar $m = 0$ sector = planar $\mathcal{N} = 4$
(M.Bershadsky,A.Johansen,hep-th/9803249)

Field content:

- Gauge group

$$U(MN') \rightarrow U_1(N') \times ... \times U_M(N')$$

- Bi-fundamental fields

$$Z \rightarrow \{Z_1, \dots Z_M\} \quad , \quad Z_I \rightarrow U_I Z_I U^\dagger_{I+1}$$

$$\Psi \rightarrow \{\Psi_1, \dots \Psi_M\} \quad , \quad \Psi_I \rightarrow U^\dagger_{I+1} \Psi_I U_I$$

M bi-fundemental chiral hypermultiplets $\left(Z_I, \bar{\Psi}_I, \chi_{ZI}, \bar{\chi}_{\Psi I}\right)$

- Adjoint fields

$$\Phi \rightarrow \{\Phi_1, \dots \Phi_M\} \quad , \quad \Phi_I \rightarrow U_I \Phi_I U^\dagger_I$$

M adjoint rep. vector multiplets $(A^\mu_I, \Phi_I, \psi_I, \psi_{\Phi I})$

Spin chain ground state with $\Delta - J = 0$

$$\mathrm{Tr}\gamma^m Z^J = \delta_{m,o}\mathrm{Tr}[(Z_1 \dots Z_M)^k] \quad , \quad J = kM$$

One-magnon state with $p_{\mathrm{mag}} = 2\pi m/M$, $J = kM$:

$$\mathrm{Tr}\gamma^m \Phi Z^{kM} = \sum_I e^{2\pi \frac{m}{M} iI} \mathrm{Tr} Z_1 \dots Z_I \Phi_I Z_{I+1} \dots Z_M (Z_1 \dots Z_M)^{k-1}$$

magnon multiplet $\Delta - J = 1$

$$\mathrm{Tr}\gamma^m D_\mu Z Z^{kM-1} \quad , \quad \mathrm{Tr}\gamma^m \Phi Z^{kM} \quad , \quad \mathrm{Tr}\gamma^m \bar{\Phi} Z^{kM}$$

$$\mathrm{Tr}\gamma^m \bar{\Psi} Z^{kM-1} \quad , \quad \mathrm{Tr}\gamma^m \Psi Z^{kM+1}$$

$SU(2) \times U(1)$ R-symmetry $\begin{pmatrix} Z \\ \bar{\Psi} \end{pmatrix}$ $\begin{pmatrix} \bar{Z} \\ \Psi \end{pmatrix}$ SU(2) doublets

$SU(2,2|2)$ supersymmetry $\rightarrow SU(2|1)^2$

Magnon limit $J \rightarrow \infty$: Since $J = kM$, either $k \rightarrow \infty$ or $M \rightarrow \infty$

enhanced supersymmetry $SU(2|2)^2 \times R^1$

String Loops:

k=1

$$\mathrm{Tr}\gamma^m \Phi Z^M = \sum_I e^{2\pi \frac{m}{M} iI} \mathrm{Tr} Z_1 ... Z_{I-1} \Phi_I Z_I ... Z_M$$

is an exact eigenstate of the full dilatation operator.

k=2

$$\left(\mathrm{Tr}\gamma^m \Phi Z^{2M}\right) \quad \pm \quad \left(\mathrm{Tr}\gamma^m \Phi Z^M\right)\left(\mathrm{Tr} Z^M\right)$$

are eigenstates with eigenvalues

$$\Delta - J = 1 + \frac{\lambda(1 \pm M/N)}{2\pi^2} \sin^2 \frac{p_{\mathrm{mag}}}{2} + ...$$

$N = MN'$

k=3

States are

$\left\{\mathrm{Tr}\gamma^m Z^{3M}, \mathrm{Tr}\gamma^m Z^{2M}\mathrm{Tr}Z^M, \mathrm{Tr}\gamma^m Z^M\mathrm{Tr}Z^{2M}, \mathrm{Tr}\gamma^m Z^M\mathrm{Tr}Z^M\mathrm{Tr}Z^M\right\}$,

eigenvalues are

$$\Delta - J = 1 + \frac{\lambda(1 \pm 2M/N)}{2\pi^2}\sin^2\frac{p_{\mathrm{mag}}}{2} + ...$$

$$\Delta - J = 1 + \frac{\lambda(1 \pm M/N)}{2\pi^2}\sin^2\frac{p_{\mathrm{mag}}}{2} + ...$$

Planar theory n'th order of perturbation theory

$\sim (g_{\mathrm{YM}}^2 N)^n(\nabla^2)^n +$ contact terms

References:

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S.Mukhi, R.Rangamani and E.Verlinde **hep-th/0204147**

String loop corrections to 1- and 2-impurity BMN states:
G.diRisi, G.Grignani, M.Orselli, G.Semenoff **hep-th/0409315**

Integrability: Bethe equations at weak coupling
B.Cheng, X.Wang and Y.S.Wu **hep-th/0403004**
P.DiVecchia, A.Tanzini **hep-th/0405262**
K.Ideguchi **hep-th/0408124**
N.Beisert, R.Roiban **hep-th/0510209**

Finite size corrections to 2-impurity states:
D.Astolfi, V.Forini, G.Grignani and G.Semenoff **hep-th/0606193**

Classical string

(Arutyunov, Frolov, Zamlakar **hep-th/0606126**)

$$S = \frac{1}{4\pi\alpha'}\int d\tau \int_{-r}^{r} d\sigma \sqrt{-h}h^{ab}\partial_a X^M G_{MN}(X)\partial_b X^N$$

$$-r \leq \sigma \leq r \quad , \quad p_M = -2\pi\alpha' G_{MN}(X)\sqrt{-h}h^{0a}\partial_a X^N$$

$$S = \frac{1}{2\pi\alpha'}\int d\tau \int_{-r}^{r} d\sigma \left[p_M \dot{X}^M + \frac{h^{01}}{h^{00}} V_1 + \frac{1}{2\sqrt{-h}h^{00}} V_2 \right]$$

$$V_1 = P_M \partial_\sigma X^M \quad , \quad V_2 = G^{MN} p_M p_N + G_{MN}\partial_\sigma X^M \partial_\sigma X^N$$

Classical string on 2-sphere $R^1 \times S^2 \subset R^1 \times S^5$

$$ds^2 = \sqrt{\lambda}\alpha' \left[-dt^2 + (1-z^2)d\phi^2 + \frac{1}{1-z^2}dz^2 \right]$$

Noether charges for translations along t and ϕ,

$$E = -\frac{\sqrt{\lambda}}{2\pi}\int_{-r}^{r} d\sigma p_t \quad , \quad J = \frac{\sqrt{\lambda}}{2\pi}\int_{-r}^{r} d\sigma p_\phi$$

Uniform light-cone gauge: parameter **a**

$$x_- = \phi - t \quad , \quad x_+ = (1-a)t + a\phi$$

$$p_- = p_\phi + p_t \quad , \quad p_+ = (1-a)p_\phi - ap_t$$

$$P_- = J - E \quad , \quad P_+ = (1-a)J + aE$$

light cone gauge (remember $\phi(\tau, r) = \phi(\tau, -r) + p_{\text{mag}}$)

$$x_+ = \tau + a\frac{p_{\text{mag}}}{2r}\sigma \quad , \quad p_+ = 1$$

Length of worldsheet $\sim P_+$

$$\frac{2\pi}{\sqrt{\lambda}}P_+ = \int_{-r}^{r} d\sigma p_+ = 2r \to r = \frac{\pi}{\sqrt{\lambda}}P_+$$

Level matching condition $\int d\sigma p_M \partial\sigma X^M = 0$

$$\Delta x_- = p_{\text{mag}} = -\int_{-r}^{r} d\sigma \left(a\frac{p_{\text{mag}}}{2r}p_- + p_z\partial_\sigma z\right)$$

solve V_1 for $\partial_\sigma x_-$, V_2 for $p_-(p_z, z, \partial_\sigma z)$

Light-cone gauge fixed action

$$S = \frac{\sqrt{\lambda}}{2\pi}\int d\tau \int_{-r}^{r} d\sigma \left[p_z \dot{z} + p_-(p_z, z, \partial_\sigma z)\right]$$

Back to configuration space $\dot{z} = -\frac{\partial p_-}{\partial p_z}$

$$S = \frac{\sqrt{\lambda}}{2\pi}\int d\tau \int_{-r}^{r} d\sigma L[z, \partial_\sigma z, \dot{z}]$$

$$L = \frac{1-(1-a)z^2}{1-2a-(1-a)^2 z^2} -$$

$$-\frac{\sqrt{(1-z^2)^2 + [1-2a-(1-a)^2 z^2][(\partial_\sigma z - a\frac{p_{\mathrm{mag}}}{2r}\dot{z})^2 - \dot{z}^2/(1-z^2)]}}{1-2a-(1-a)^2 z^2}$$

giant magnon is a right-moving soliton

$$z(\tau,\sigma) = z(\sigma - v\tau - av\frac{p_{\mathrm{mag}}}{2r}\sigma) \quad , \quad \partial_\sigma z - a\frac{p_{\mathrm{mag}}}{2r}\dot{z} = z'$$

$$L = \frac{1-(1-a)z^2}{1-2a-(1-a)^2 z^2} -$$

$$- \frac{\sqrt{(1-z^2)^2 + [1-2a-(1-a)^2 z^2][(1-v^2/(1-z^2))(z')^2]}}{1-2a-(1-a)^2 z^2}$$

"conservation of energy" $\frac{\omega-1}{1-a+a\omega} = z' \frac{\partial L}{\partial z'} - L$

turning points: $z^2_{\rm max} = 1 - v^2$, $z^2_{\rm min} = 1 - 1/\omega^2$

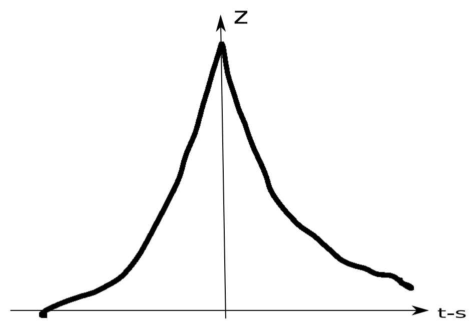
$$\frac{1}{|z'|} = \left[1 - a + \frac{a}{\omega(1-z^2)}\right] \sqrt{\frac{z^2_{\rm max} - z^2}{z^2 - z^2_{\rm min}}}$$

solution:

$$\sigma - v\tau - av\frac{p_{\rm mag}}{2r}\sigma = \int_{z_{\rm max}}^{z} dz \left[1 - a + \frac{a}{\omega(1-z^2)}\right]\sqrt{\frac{z_{\rm max}^2 - z^2}{z^2 - z_{\rm min}^2}}$$

$$\frac{\sigma - v\tau}{\sqrt{1-v^2}} = (F - E)\left(\sin^{-1}\sqrt{\frac{1 - z^2/z_{\rm max}^2}{1 - z_{\rm min}^2/z_{\rm max}^2}}, \sqrt{1 - z_{\rm min}^2/z_{\rm max}^2}\right)$$

$F(\phi, k)$ and $E(\phi, k)$ are incomplete elliptic functions of 1'st,2nd kind



In the conformal gauge

$$\frac{\sigma - v\tau}{\sqrt{1-v^2}} = F\left(\sin^{-1}\sqrt{\frac{1 - z^2/z_{\rm max}^2}{1 - z_{\rm min}^2/z_{\rm max}^2}}, \sqrt{1 - z_{\rm min}^2/z_{\rm max}^2}\right)$$

Integrate to find observables:

$$\frac{1}{|z'|} = \left[1 - a + \frac{a}{\omega(1-z^2)}\right]\sqrt{\frac{z_{\rm max}^2 - z^2}{z^2 - z_{\rm min}^2}}$$

$$\frac{\pi}{\sqrt{\lambda}}[J + (E - J)] = r = \int_0^r d\sigma = \frac{1}{1 - avp_{\rm mag}/2r}\int_{z_{\rm min}}^{z_{\rm max}} \frac{dz}{|z'|}$$

$$E - J = -\frac{\sqrt{\lambda}}{2\pi}\int_{-r}^{r} d\sigma p_z \partial_\sigma z = -\frac{\sqrt{\lambda}}{\pi}\int_{z_{\rm min}}^{z_{\rm max}} dz p_z$$

$$p_{\rm mag} = -\frac{2ap_{\rm mag}}{2r - ap_{\rm mag}v}\int_{z_{\rm min}}^{z_{\rm max}} dz\frac{p_-}{|z'|} + 2\int_{z_{\rm min}}^{z_{\rm max}} dz|p_z|$$

complicated dependence on gauge paramter **a**

non-linear $r = \frac{\pi}{\sqrt{\lambda}}P_+$

can solve for observables and the solution is a-independent

Result: $\quad , \ \eta = \frac{1/\omega^2 - v^2}{1-v^2} = \frac{z_{\rm max}^2 - z_{\rm min}^2}{z_{\rm max}^2}$

$$
\begin{aligned}
E - J &= \frac{\sqrt{\lambda}}{\pi}\left[\sqrt{1-v^2}\,(K(\eta) - E(\eta))\right] \\
J &= \frac{\sqrt{\lambda}}{\pi}\left[\sqrt{1-v^2}E(\eta) - (\omega - 1)\frac{(1+\omega v^2)K(\eta)}{\omega\sqrt{1-v^2}}\right] \\
p_{\rm mag} &= -2\frac{\omega^2 v^2 K(\eta) - \Pi(\eta - \eta/v^2, \eta)}{v\omega\sqrt{1-v^2}}
\end{aligned}
$$

E, K, Π are complete elliptic functions of 1'st,2nd,3rd kind.
Right-hand-side depends of v, ω. Eliminate these in terms of J and $p_{\rm mag}$, then obtain the equation for the spectrum: $E - J$ as a function of J, $p_{\rm mag}$
can be done explicitly when J is large

Conclusions:

- Finite size corrections are computable by asymptotic expansion in J

$$E-J = \frac{\sqrt{\lambda}}{\pi}\left|\sin\frac{p_{\rm mag}}{2}\right|\left[1-4\sin^2\frac{p_{\rm mag}}{2}e^{-2-2\pi\frac{J}{\sqrt{\lambda}|\sin p_{\rm mag}/2|}}+\ldots\right]$$

orbifold $p_{\rm mag} = 2\pi m/M$

- Can be interpreted as magnon of $\mathcal{N}=2$ quiver SYM.
- Where would exponentially small finite size corrections arise on the gauge theory side? Wrapping interactions.
- Quantization of the string sigma model – asymptotic expansion in $1/\sqrt{\lambda}$ about the giant magnon is also possible.