Conformal properties of four-gluon amplitudes in $\mathcal{N} = 4$ super-Yang-Mills

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Based on work in collaboration with

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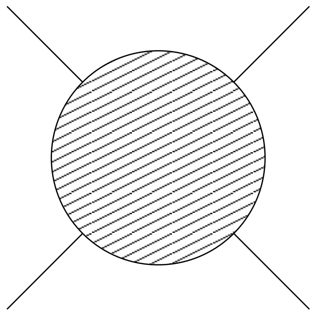
I. Gluon amplitudes in perturbation theory

S-matrix elements for four-gluon scattering are defined with all external legs on shell, $p^2_{1,2,3,4} = 0 \Rightarrow$ IR singularities.



Although the amplitude is not a good observable itself, it contains important information about the cusp anomalous dimension $\gamma_{\rm cusp}$ (the large spin limit of the anomalous dimension of twist-two operators). Recent results in the planar limit allowed comparison with predictions from integrable models (BES equation). An interesting iterative structure was discovered revealing properties valid to all orders – relevant to AdS/CFT.

The subject of this talk will be the conformal symmetry underlying the amplitude. We first review the perturbative calculations of Bern et al and discuss the conformal properties of the Feynman integrals. Then we establish the exact form of the IR finite part of the amplitude using factorization into form-factors and imposing conformal invariance.

II. Loops: early results

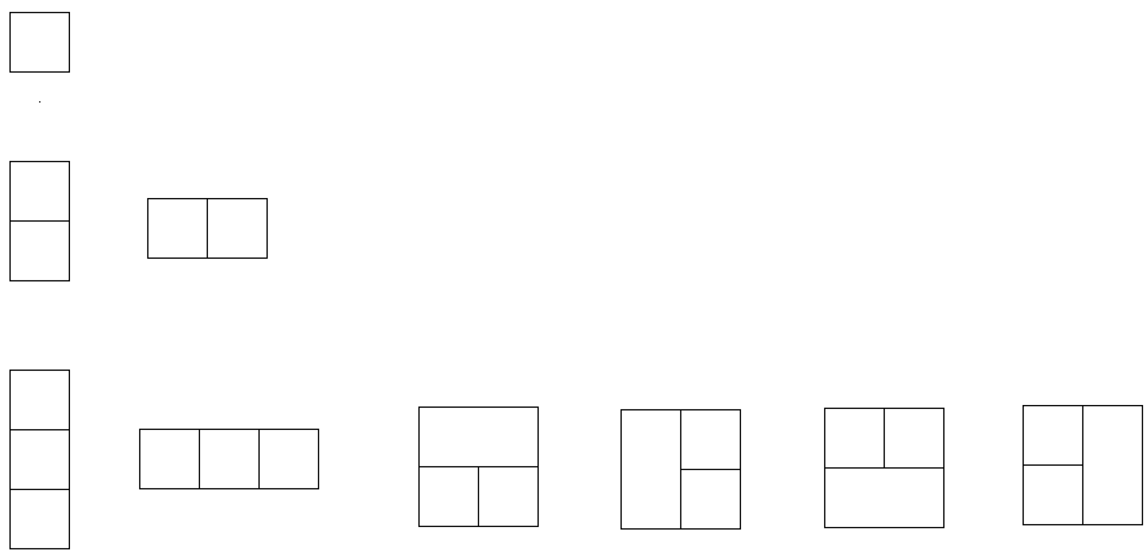
• 1-loop: Green, Schwarz & Brink (1982); Gates, Grisaru, Rocek & Siegel (1983)

• 2-loop: Bern, Rozowsky & Yan (1997); all-order iteration conjectured Anastasiou, Bern, Dixon & Kosower (2003)

• 3-loop: Bern, Dixon & Smirnov (2005) : iteration confirmed; found $\gamma_{\rm cusp}$ which matches the maximal transcedentality prediction of Kotikov, Lipatov, Onischenko & Velizhanin (2004) based on QCD results by Moch, Vermaseren & Vogt (2003), and confirmed by Staudacher (2004) using Bethe Ansatz

The unitarity method employed by Bern et al is much more efficient than a direct calculation of Feynman graphs. It consists in constructing the amplitude from a remarkably simple set of Feynman integrals and then verifying its consistency through unitarity cuts.

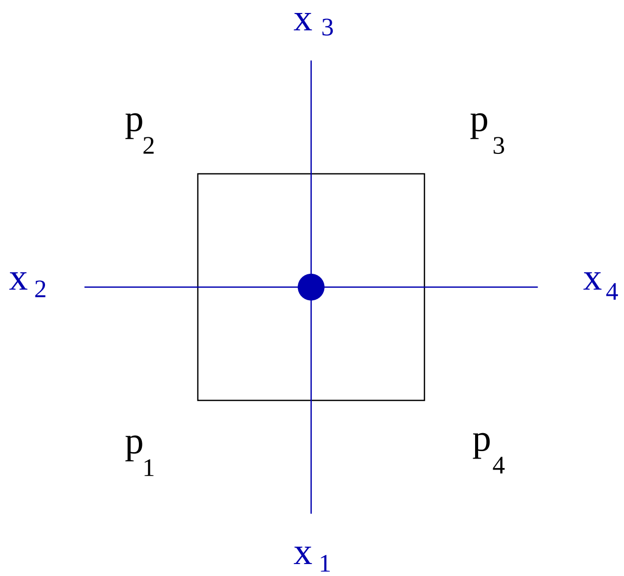
Feynman integrals up to 3 loops:



III. Conformal properties of 4-point loop integrals

Ladder integrals (scalar boxes) are known to be conformal in $D = 4$ (i.e., off shell) (Broadhurst (1993)).

Drummond, Henn, Smirnov & Sokatchev (2006) realized that also the 'tennis court' 3-loop integral of BDS is conformal. The best way to see this is to go to a dual 'coordinate space' picture (not a Fourier transform !):



Momentum integral off shell ($p_i^2 \neq 0,\ D = 4$)

$$\int \frac{d^4k}{k^2(k-p_1)^2(k-p_1-p_2)^2(k+p_4)^2} \Rightarrow$$

Change variables:

$$p_1 = x_1 - x_2 \equiv x_{12},\ p_2 = x_{23},\ p_3 = x_{34},\ p_4 = x_{41}\,, \qquad \sum_i p_i = 0$$

$$k = x_{15}$$

$$\Rightarrow \int \frac{d^4x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} = \frac{-i\pi^2}{x_{13}^2 x_{24}^2}\Phi^{(1)}(u,v)$$

Function of the conformal invariant cross-ratios

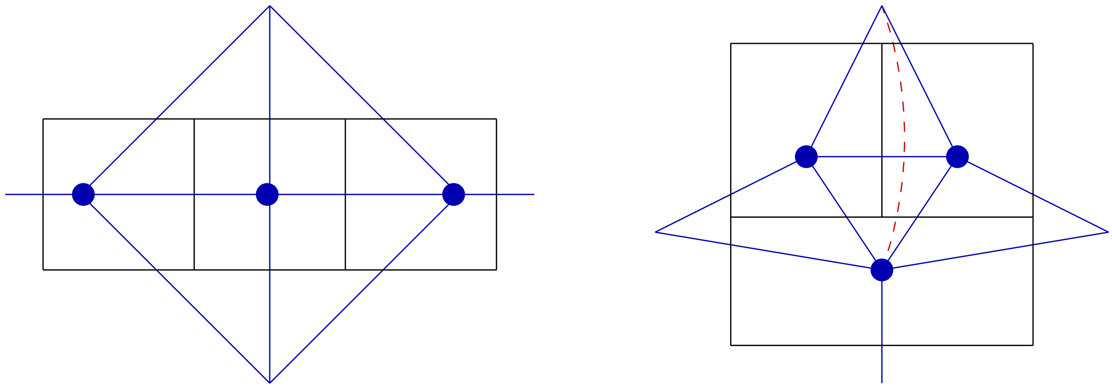
$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

Check conformal invariance by inversion

$$x \longrightarrow \frac{x}{x^2}$$

Need 4 scalar propagators at each 4-dimensional integration vertex to cancel the conformal weight of the measure.

Further examples: 3-loop integrals:

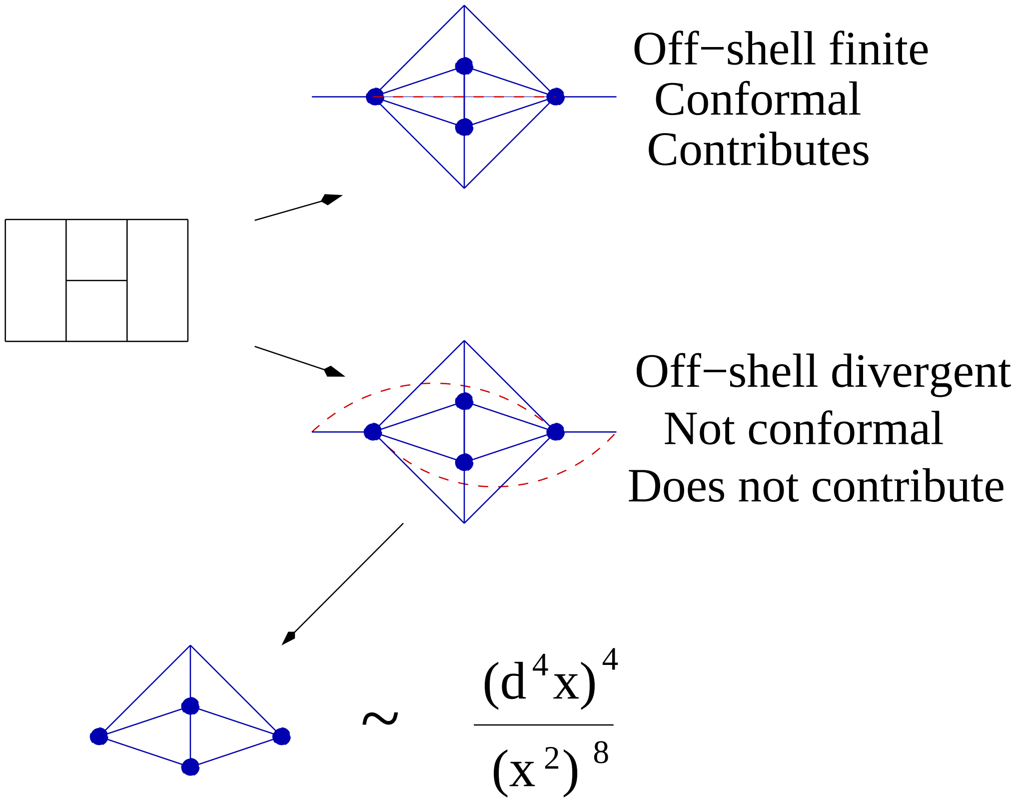


The 'tennis court' requires a numerator factor x^2 to balance the excess of propagators at one vertex.

The same pattern continues at higher loops.

Bern, Czakon, Dixon, Kosower & Smirnov (2006) used this observation to list all relevant 4-loop integrals. They found 10 integrals, but unitarity shows that 2 of them do not contribute. Why?

We can give a simple conformal explanation: the non-contributing integrals diverge even off shell. Example:



Exactly the same happens at 5 loops (Bern, Carrasco, Johansson & Kosower (2007)). They find 34 contributing and 25 non-contributing integrals. We have checked that the latter diverge.

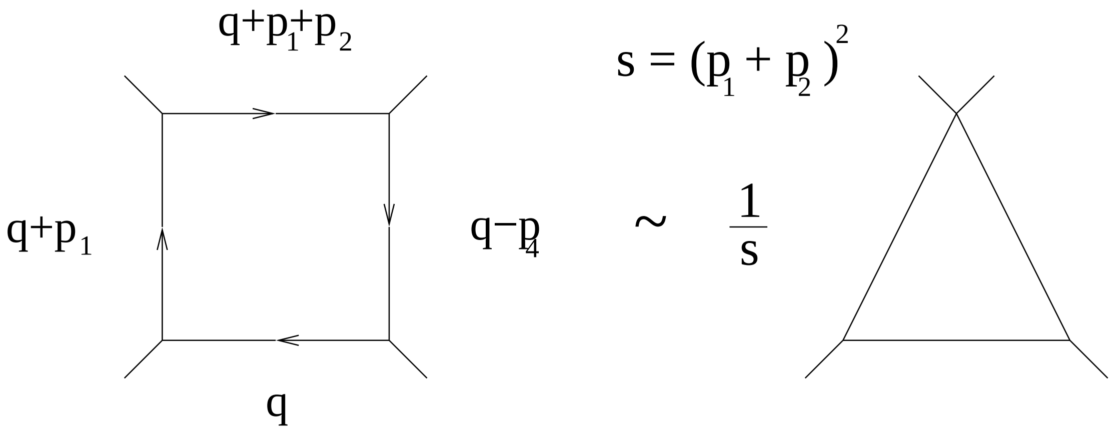
Another remarkable feature is that the contributing integrals always come with coefficients ± 1 (still to be understood?):

$$\mathcal{A} = \sum \pm \text{ All planar conformal integrals}$$

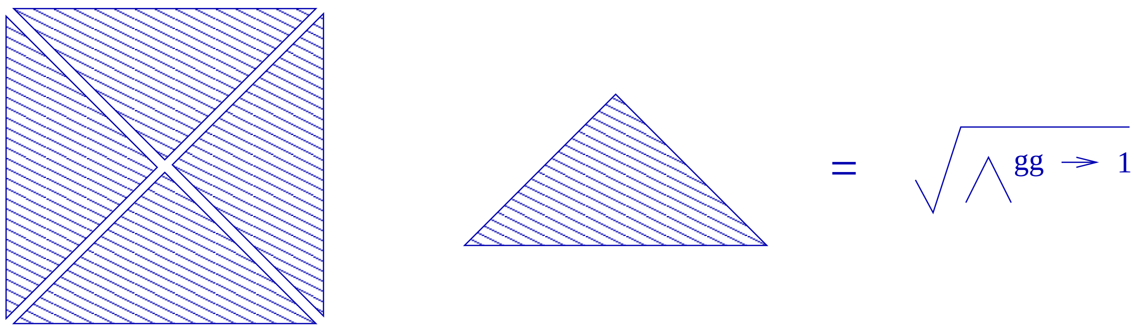
Strong evidence for an underlying conformal structure – does not trivially follow from the fact that the theory is conformal!

IV. Factorization and exponentiation. Finite part and conformal invariance

IR singularities of on-shell amplitudes are determined by Sudakov's form-factor, i.e. the IR singular part of the amplitude factorizes into 3-point factors. Simple example:



Generalization to higher loops:



IR singularities are due to exchanges of soft gluons between neighboring hard lines. Trivial color structure in the planar limit.

Exponentiation of form-factors – two typical regimes:

• on-shell: $p_1^2 = p_2^2 = 0$, need an IR regulator, e.g. dimensional $D = 4 - 2\epsilon$, $\epsilon < 0$ and mass scale μ

• off-shell: $p_1^2 = p_2^2 = m^2$, $m^2 << s = (p_1 + p_2)^2$ (Mandelstam variable); m^2 is the IR cutoff, $D = 4$

On-shell exponentiation

The on-shell form-factor satisfies an evolution equation (Collins (1980), Mueller (1989), Sen (1989, Korchemsky (1989), Magnea & Sterman (1990)) whose solution is very simple in a finite theory ($\beta = 0$, no running of the coupling)

$$\ln \Lambda^{gg\to 1} = -\frac{1}{4}\sum_{l=1}^{\infty} \alpha^l \left(\frac{\mu^2}{s}\right)^{l\epsilon} \left[\frac{\gamma^{(l)}_{\rm cusp}}{(l\epsilon)^2} + \frac{G^{(l)}}{l\epsilon}\right] + \text{finite}$$

where $\alpha = \frac{g^2 N}{8\pi^2}$ is the 't Hooft coupling.

Combining two form-factors in the $s = (p_1 + p_2)^2$ and $t = (p_2 + p_3)^2$ channels, the 4-gluon amplitude factorizes into

$$M_4 = (\text{finite}) \;\times\; \Lambda^{gg\to 1}(s) \;\times\; \Lambda^{gg\to 1}(t)$$

Compare this to the amplitude of Bern, Dixon & Smirnov:

$$\ln M_4^{BDS} \;=\; \text{Pole part}\left(\frac{\mu^2}{s}\right) + \text{Pole part}\left(\frac{\mu^2}{t}\right) + \frac{\gamma_{\rm cusp}(\alpha)}{8}\ln^2\frac{s}{t} + \text{const}(\alpha) + O(\epsilon)$$

The special form of the finite part is the main observation of BDS.

Recently Alday & Maldacena were able, using string theory considerations, to reproduce the same factorized form of the amplitude at strong coupling. For them $\ln M_4$ is proportional to the area of the worldsheet of a classical string propagating in AdS space. They use precisely our change of variables $p_i = x_{i,i+1}$ (called T-duality) and look for a minimal surface ending on the curve of light-like segments $x_{i,i+1}$ (Wilson loop?).

Off-shell exponentiation and conformal invariance

Using an off-shell cutoff $p^2_{1,2,3,4} = m^2$ allows us to stay in $D = 4$ and reveal the conformal structure. The form-factor exponentiates as follows (if $\beta = 0$) (Korchemsky (1989))

$$\ln \Lambda^{gg\to 1} = -\frac{\gamma_{\rm cusp}(\alpha)}{4} \ln^2\left(\frac{m^2}{s}\right) + C(\alpha) \ln\left(\frac{m^2}{s}\right) + \text{finite}$$

Factorization of the 4-point amplitude:

$$\begin{aligned} \ln M_4^{\rm off-shell} &= \ln \Lambda^{gg\to 1}\left(\frac{m^2}{s}\right) + \ln \Lambda^{gg\to 1}\left(\frac{m^2}{t}\right) + \text{finite} \\ &= -\frac{\gamma_{\rm cusp}(\alpha)}{8}\left[\ln^2\left(\frac{m^4}{st}\right) + \ln^2\frac{s}{t}\right] \\ &+ C(\alpha) \ln\left(\frac{m^4}{st}\right) + \text{Finite part}\,(m^2, s, t) \end{aligned}$$

Recall the conformal cross-ratios

$$u = \frac{x^2_{12}x^2_{34}}{x^2_{13}x^2_{24}}, \qquad v = \frac{x^2_{14}x^2_{23}}{x^2_{13}x^2_{24}}$$

Now we have $p_i^2 = x^2_{i,i+1} = m^2$, $x^2_{13}x^2_{24} = st$, hence

$$u = v = \frac{m^4}{st}$$

However, $\frac{s}{t}$ is not a conformal invariant ! $M_4^{\rm off-shell}$ becomes conformal if $\ln^2\frac{s}{t}$ is compensated by the finite part:

$$\text{Finite part} = \frac{\gamma_{\rm cusp}(\alpha)}{8}\ln^2\frac{s}{t} + \text{const}(\alpha) + O\left(\frac{m^4}{st}\right)$$

exactly as in M_4^{BDS} and at strong coupling!

V. Conclusion

Main points:

• The four-gluon amplitude in $\mathcal{N} = 4$ SYM has the form

$$\mathcal{A} = \sum \pm \text{ All planar conformal integrals}$$

The integrals must be well defined off shell. Need to explain the coefficients ± 1.

• Basic properties of IR divergences in field theory together with the assumption of off-shell conformal invariance determine the form of the IR finite part of the amplitude. The argument holds both in perturbation theory and at strong coupling, in agreement with the observations of Bern et al and Alday & Maldacena.

We have seen overwhelming evidence for conformal invariance in the 4-gluon amplitude in $\mathcal{N} = 4$ SYM. More effort is needed to understand its origin and exploit its consequence (work in progress in collaboration with Brandhuber, Drummond, Henn, Heslop, Korchemsky, Travaglini).