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Origin of dressing phase in $\mathcal{N} = 4$ Super Yang-Mills

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hep-th/0703177, also work in progress

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1. Introduction

AdS/CFT correspondence --- a gauge/string duality

(Maldacena '97, Gubser-Klebanov-Polyakov '97, Witten '97)

planar $\mathcal{N} = 4$ U(N) super Yang-Mills

$\Updownarrow$

- gluon dynamics common to QCDs

free IIB superstrings on $AdS_5 \times S^5$

- superstrings in the simplest curved background

- We consider $N \to \infty$ limit $\Rightarrow$ integrability strong/weak correspondence

$\longrightarrow \lambda$

$\lambda \ll 1$ $\lambda \to \infty$

super Yang-Mills at one-loop $\Rightarrow$ higher loops

quantum strings $\Leftarrow$ classical strings on $AdS_5 \times S^5$

$\Downarrow$

conventional spin-chain

integrable

(Minahan-Zarembo '02)
(Beisert-Staudacher '03)
(Beisert '03)

$\Downarrow$

sigma-model on a coset space

integrable

(Bena-Roiban-Polchinski '03)

(assuming that) integrable at general λ

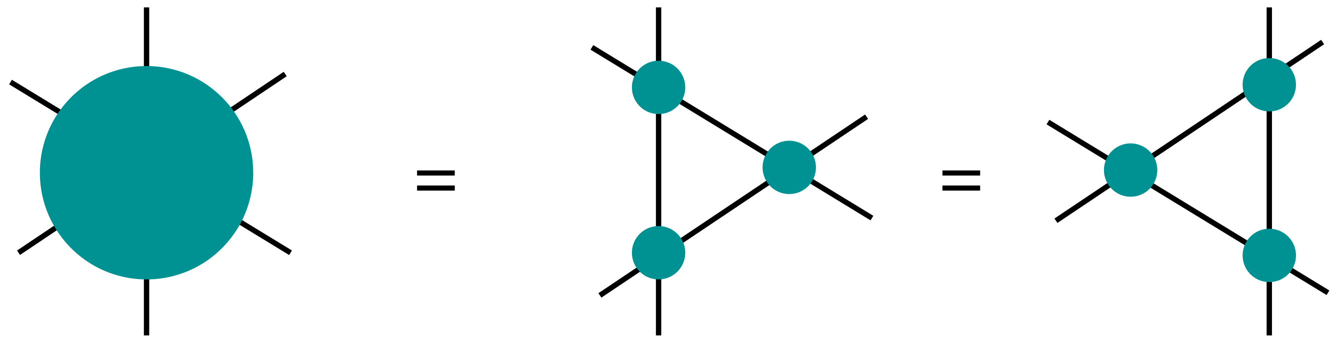
particle model

$(L \to \infty)$

(Staudacher '04)
(Beisert '05)

Integrability of 2D particle models

- Factorization of multi-particle scattering amplitudes



$$\left\{ \begin{array}{l} \text{dispersion relation } \boldsymbol{E(p)} \text{ for 1 particle} \\ \text{scattering matrix } \boldsymbol{\hat{S}(p_1, p_2)} \text{ for 2 particles} \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} \text{all scattering amplitudes} \\ \text{spectra of conserved charges} \end{array} \right\} \text{ are determined}$$

The present particle model

(Berenstein-Maldacena-Nastase '02)

(Staudacher '04) (Beisert '05, '06)

- Vacuum

$$|0\rangle^{\mathrm{I}} := |\cdots ZZZZZZZZZZZZZZZZZZ\cdots\rangle$$

- Asymptotic state

$$|X_1 X_2\rangle^{\mathrm{I}}$$

$$:= \sum_{n_1 \ll n_2} e^{ip_1 n_1 + ip_2 n_2} |\cdots ZZZ \underset{\uparrow n_1}{X_1} ZZZ \cdots ZZZ \underset{\uparrow n_2}{X_2} ZZZ \cdots\rangle$$

- One particle states: 8 bosons + 8 fermions

$$\Phi_i \ {}_{(i=1,\dots,4)}, \quad D_i Z \ {}_{(i=1,\dots,4)}, \quad \Psi_{\alpha\dot{a}}, \Psi_{a\dot{\alpha}} \ {}_{(a,\alpha=1,\dots,2)}$$

: single excitation of Z

$$\bar{Z}, F_{\alpha\beta}, D_i \Phi_j, \dots$$

: multiple excitation

- Spontaneous breaking of the global symmetry

$$\begin{array}{ccccc} PSU(2,2|4) & \rightarrow & PSU(2|2) \times PSU(2|2) \ltimes \mathbb{R} \\ (8|8) & = & (2|2) \quad \times \quad (2|2) \end{array}$$

Z	ϕ_1	ϕ_2	ψ_1	ψ_2
$\bar{\phi}_1$	Φ_{11}	Φ_{12}	Ψ_{11}	Ψ_{12}
$\bar{\phi}_2$	Φ_{21}	Φ_{22}	Ψ_{21}	Ψ_{22}
$\bar{\psi}_1$	$\dot{\Psi}_{11}$	$\dot{\Psi}_{12}$	$D_{11}Z$	$D_{12}Z$
$\bar{\psi}_2$	$\dot{\Psi}_{21}$	$\dot{\Psi}_{22}$	$D_{21}Z$	$D_{22}Z$

Characteristic of the particle model: (Beisert's talk)

$PSU(2|2) \times PSU(2|2) \ltimes \mathbb{R}^3$ centrally extended symmetry

Strings on $AdS_5 \times S^5$ in the uniform light-cone gauge

$\Longrightarrow$ particle model with the same symmetry (Arutyunov, Frolov, Plefka, Zamaklar '05, '06)

- The symmetry fully determines
  - dispersion relation (Beisert '05)
  - S-matrix up to an overall scalar factor

- Determination of the remaining scalar factor

(Arutyunov-Frolov-Staudacher '04, Janik '06, Hernández-López '06, Beisert-Hernández-López '06, ...)

- Closed integral formula

(Beisert-Eden-Staudacher '06) (see also Dorey-Hofman-Maldacena '07)

Determination of the scalar factor

A) Solving physical constraints (on-shell formulation)

crossing symmetry, poles and branch cuts,
perturbative computation, etc.

(Zamolodchikov^2 '77)

(Arutyunov-Frolov-Staudacher '04, Janik '06, Hernández-López '06,
Beisert-Hernández-López '06, Beisert-Eden-Staudacher '06, ...)

B) Direct computation (off-shell formulation)

effective phase of underlying bare integrable model

(Korepin '79, Faddeev-Takhtajan '81, Andrei-Destri '84)

(KS-Satoh '07)

A) On-shell formulation

Zamolodchikovs' derivation

i) Lie algebra and its representation

$$\hat{R}(u) = c_1(u)\hat{I} + c_2(u)\hat{P}$$

ii) unitarity, associativity (=Yang-Baxter Eqs.)

$$\hat{R}(u) = \frac{u}{u+i}\hat{I} + \frac{i}{u+i}\hat{P}$$

iii) crossing symmetry

$$\hat{S}(u) = X_{\mathrm{CDD}}(u) S_0(u) \hat{R}(u)$$

iv) pole analysis

$$\hat{S}(u) = S_0(u) \hat{R}(u)$$

AdS/CFT particle model

(i) String side

Mismatch between all-loop Bethe eqs. and classical strings

"three-loop discrepancy"

It can be repaired by a dressing factor in the Bethe eqs.

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{K} \sigma^2(u_j, u_k) \frac{u_k - u_j + i}{u_k - u_j - i}$$

AFS factor (Arutyunov-Frolov-Staudacher '04)

scalar factor of the S-matrix = dressing factor in the Bethe eqs.

AFS factor: correct at the leading semi-classical order

Quantum corrections in the worldsheet theory : $\frac{1}{\sqrt{\lambda}}$ expansion

(Hernández-López '06)

(Freyhult-Kristjansen '06)

(Gromov-Vieira '07)

All order conjecture

(Beisert-Hernández-López '06)

consistent with

(Arutyunov-Frolov '06)

Crossing symmetry

(Janik '06)

(ii) Gauge theory side

Low twist operators

$$\mathcal{O} = \mathrm{Tr}(D^S Z^L) + \cdots \qquad\qquad S \gg L(= 2, 3, \ldots)$$

soft(cusp) anomalous dimension:

$$\Delta = S + f(g) \log S + \mathcal{O}(S^0)$$

$f(g)$: universal scaling function

$\uparrow\downarrow$ (Eden-Staudacher '06) (Beisert-Eden-Staudacher '06)

$S_0(p_1, p_2; g)$: scalar factor

trivial up to three loops

Proposal of Beisert-Eden-Staudacher

1/2 of the expected phase

- Based on phenomenological principles:

$$\left\{\begin{array}{ll}\text{scaling law} & \text{(Beisert-Klose '05)}\\ \text{transcendentality} & \text{(Kotikov-Lipatov '02)}\\ \text{cancellation of } \zeta(2n+1) & \end{array}\right\} \Longrightarrow$$

A phase factor is uniquely fixed order by order

(cf. Eden's talk)

- "Analytic continuation" from the string side
- Numerical tests against MHV amplitudes at 4 loops

(Bern-Czakon-Dixon-Kosower-Smirnov '06) (Cachazo-Spradlin-Volovich '06)

Closed integral formula (in the Fourier space)

$$\hat{K}_d(t,t') = 8g^2 \int_0^\infty dt'' \hat{K}_1(t, 2gt'') \frac{t''}{e^{t''}-1} \hat{K}_0(2gt'', t)$$

$$\hat{K}_0(t,t') = \frac{tJ_1(t)J_0(t') - t'J_0(t)J_1(t')}{t^2 - t'^2} \qquad \hat{K}_1(t,t') = \frac{t'J_1(t)J_0(t') - tJ_0(t)J_1(t')}{t^2 - t'^2}$$

- How to understand its structural simplicity?
- Any simple derivation/interpretation?

Emergence of such integral kernels
in solving the nested levels of Bethe eqs.

(Rej-Staudacher-Zieme '07)

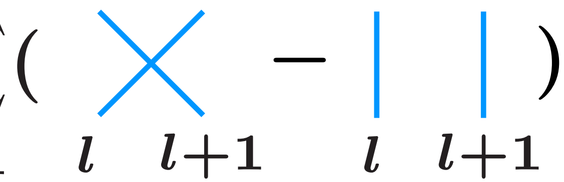
B) Direct computation

su(2) R-matrix

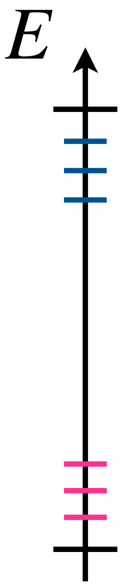
⇨ BAE for Heisenberg spin-chain

$$\left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right)^L=\prod_{l\neq k}^{J}\frac{u_k-u_l+i}{u_k-u_l-i}$$

$H = \sum_{l=1}^{L}$



anti-ferromagnetic chain



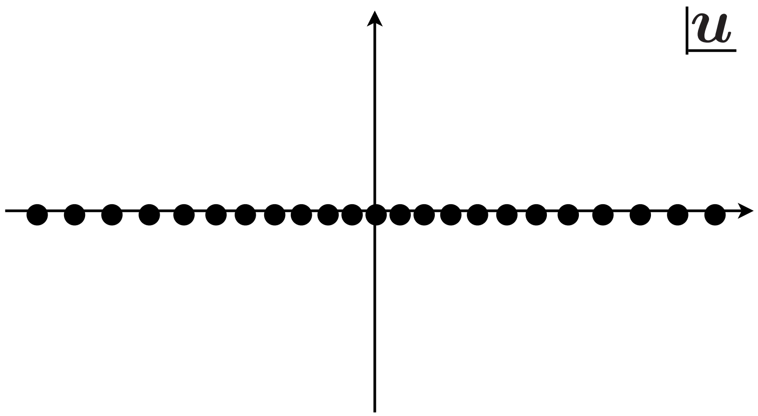
ferromagnetic state

magnon states

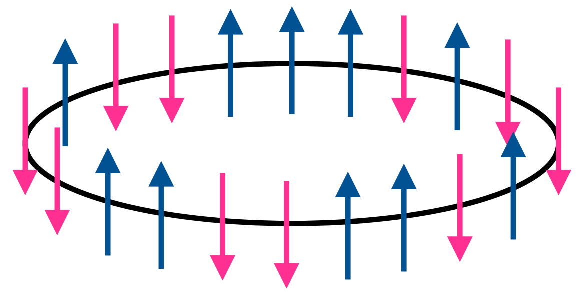
spinon states ⇨ fundamental excitations

anti-ferromagnetic state ⇨ vacuum

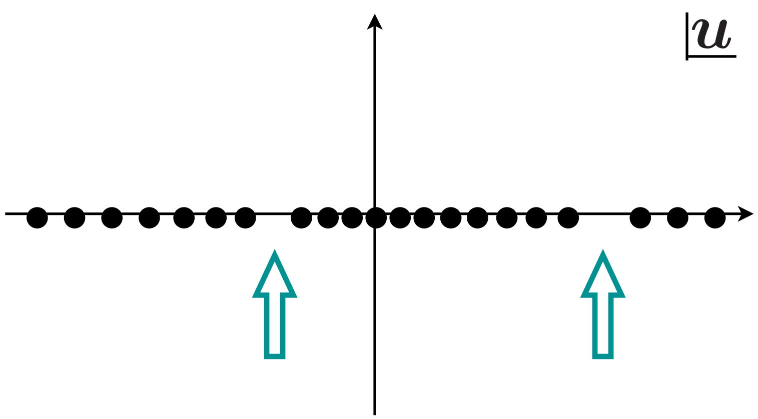
antiferromagnetic ground states



$\Longleftrightarrow \sum$



2-spinon excitated states: 2-holes



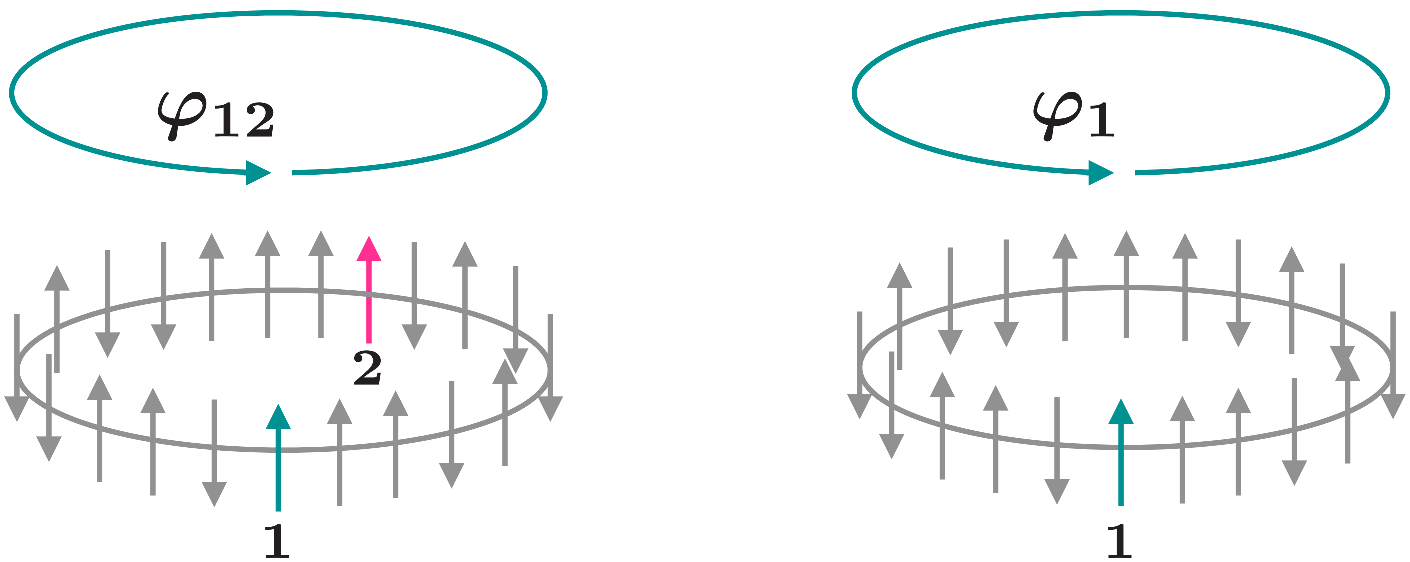
scattering phase of the 2-spinons

$$\Longrightarrow \quad S_0(u) = i\frac{\Gamma(-\frac{u}{2i})\Gamma(\frac{1}{2}+\frac{u}{2i})}{\Gamma(\frac{u}{2i})\Gamma(\frac{1}{2}-\frac{u}{2i})}$$

Scalar factor of the Zamolodchikovs' S-matrix

How to compute the scattering phase?

Total scattering phase that the particle 1 acquires
in the presence/absence of the particle 2



$$\varphi_{12} - \varphi_1 = \delta_{\text{bare}} + \delta_{\text{back-reaction}}$$

⇧ ⇩

ln (R-matrix) $\ln S_0$

- $\mathfrak{su}(2)$ R-matrix

  $\Longrightarrow$ Bethe equations for the $\mathfrak{su}(2)$ Heisenberg spin-chain

antiferromagnetic vacuum $\sim \uparrow\downarrow$

  $\Longrightarrow$ Zamolodchikovs' S-matrix

- $\mathfrak{su}(2|2) \otimes \mathfrak{su}(2|2)$ R-matrix

  $\Longrightarrow$ Asymptotic all-loop $\mathfrak{psu}(2,2|4)$ Bethe equations (without the dressing phase)

  (Beisert-Staudacher '05)

"antiferromagnetic" vacuum $\sim |\phi_1 \phi_2 Z^+\rangle + |\psi_1 \psi_2\rangle$

  $\Longrightarrow$ $\mathfrak{su}(2|2) \otimes \mathfrak{su}(2|2)$ S-matrix with the dressing factor

  (KS-Satoh '07) (cf. Rej-Staudacher-Zieme '07)

Starting point: $\mathfrak{su}(2|2) \otimes \mathfrak{su}(2|2)$ R-matrix

(S-matrix without dressing factor)

$$\mathcal{S} = \underset{\underset{\mathbf{1}}{\parallel}}{\underline{S_0}} \left[\mathcal{R}_{\mathfrak{su}(2|2)} \otimes \mathcal{R}_{\mathfrak{su}(2|2)} \right]$$

- Periodic boundary condition

Yang equations: $$e^{ip_l L} = \prod_{j \neq l}^{K_4} \mathcal{S}(p_l, p_j)$$

$\Downarrow$ diagonalization (by nested Bethe ansatz)

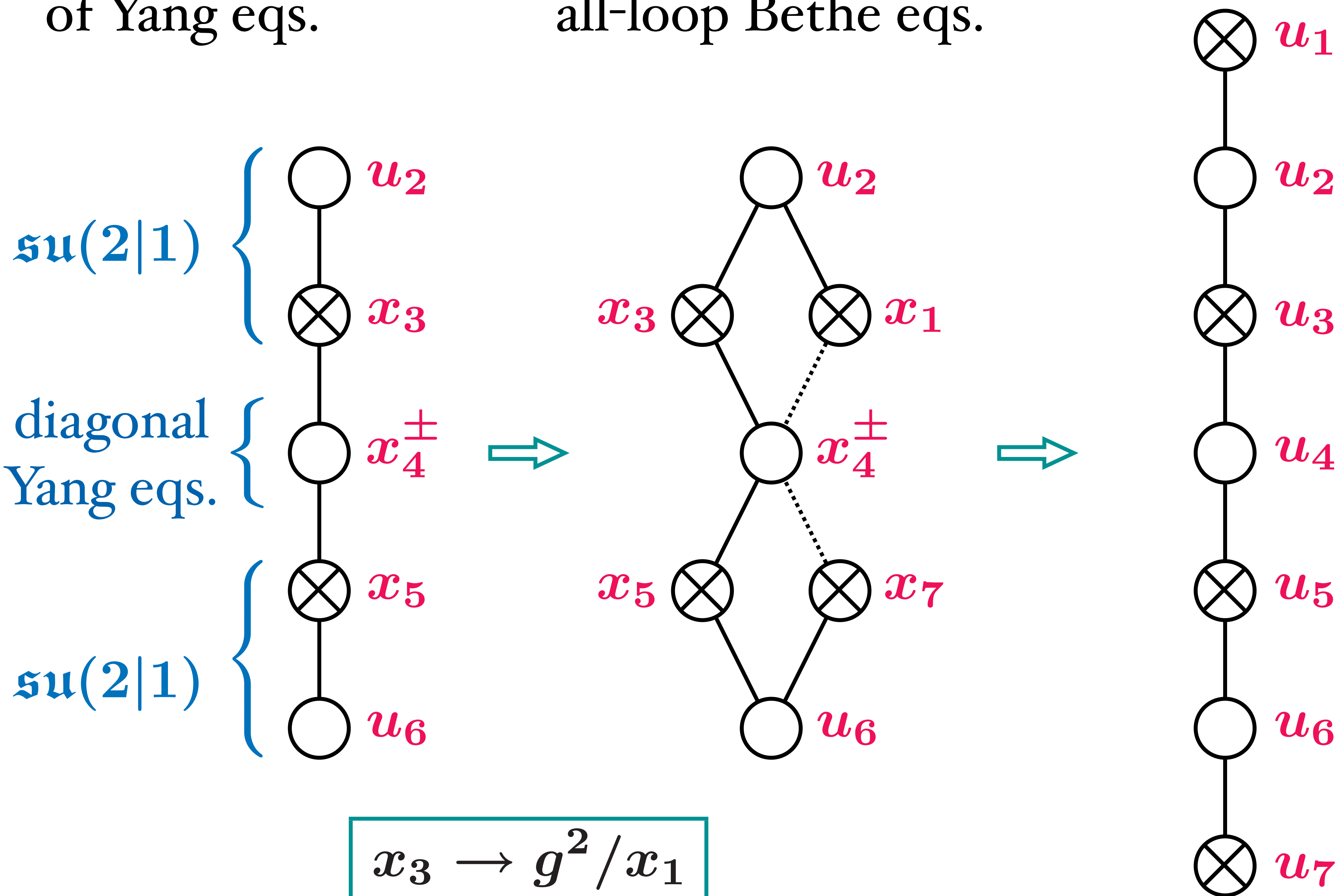
(Beisert '05, Martins-Melo '07, de Leeuw '07, ...)

Asymptotic all-loop $\mathfrak{psu}(2,2|4)$ Bethe ansätze

(Here: no direct correspondence with Yang-Mills operators) diagonalization of Yang eqs.

Beisert-Staudacher all-loop Bethe eqs.

one-loop



$$x_3 \longrightarrow g^2/x_1$$
$$x_5 \longrightarrow g^2/x_7$$

$$g \longrightarrow 0$$

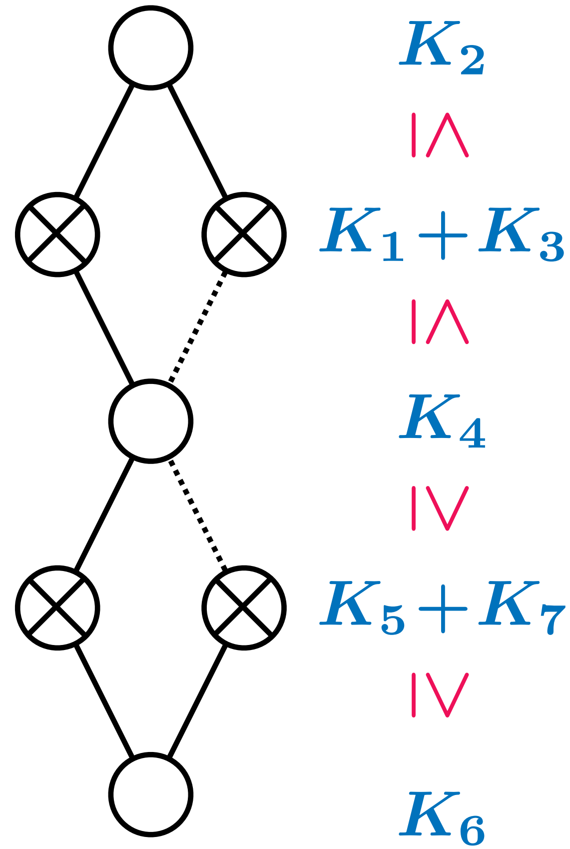
Rapidity variables

$$x^{\pm}(u) = x(u \pm \tfrac{i}{2})$$

$$x(u) = \frac{u}{2}\left(1 + \sqrt{1 - 4g^2/u^2}\right)$$

$$g = \frac{\sqrt{\lambda}}{4\pi}$$

Constraints on the occupation numbers



How to construct the "anti-ferromagnetic" vacuum?

$$\prod_{j=1}^{K_4} \frac{1 - g^2/x_{7,l}\, x_{4,j}^+}{1 - g^2/x_{7,l}\, x_{4,j}^-} = \prod_{j=1}^{K_6} \frac{u_{7,l} - u_{6,j} - i/2}{u_{7,l} - u_{6,j} + i/2}$$

$$\prod_{j=1}^{K_7} \frac{u_{6,l} - u_{7,j} + i/2}{u_{6,l} - u_{7,j} - i/2} = \prod_{j\neq l}^{K_6} \frac{u_{6,l} - u_{6,j} + i}{u_{6,l} - u_{6,j} - i}$$

$$\wr\wr$$

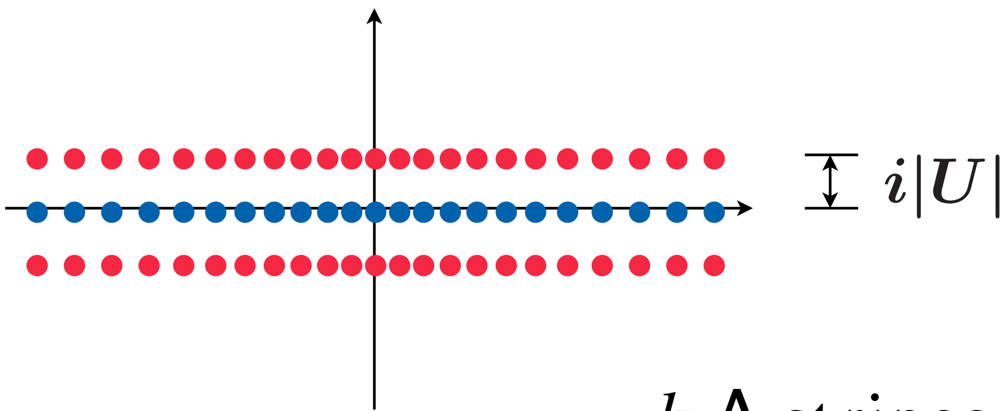
$$e^{ik_l L_H} = \prod_{j=1}^{M} \frac{\sin k_l - \Lambda_j - i|U|}{\sin k_l - \Lambda_j + i|U|}$$

$$\prod_{j=1}^{N_e} \frac{\Lambda_l - \sin k_j + i|U|}{\Lambda_l - \sin k_j - i|U|} = \prod_{j\neq l}^{M} \frac{\Lambda_l - \Lambda_j + 2i|U|}{\Lambda_l - \Lambda_j - 2i|U|}$$

Lieb-Wu equations for the Hubbard model in the attractive case $(U < 0)$

Ground state configuration

(Woynarovich '83, Essler-Korepin '94)

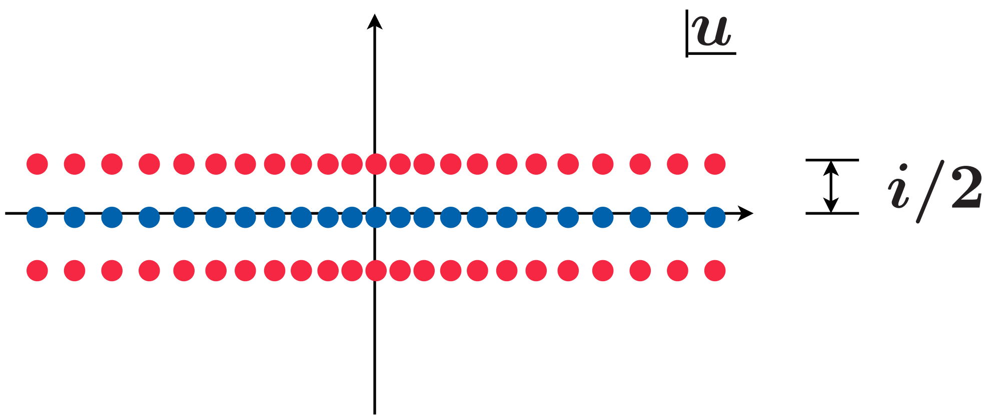


- Λ
- $\sin k$

Hubbard model (attractive case)

k-Λ strings

$\Downarrow$



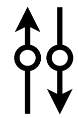
- u_6
- u_7

'AF vacuum' of the bare model

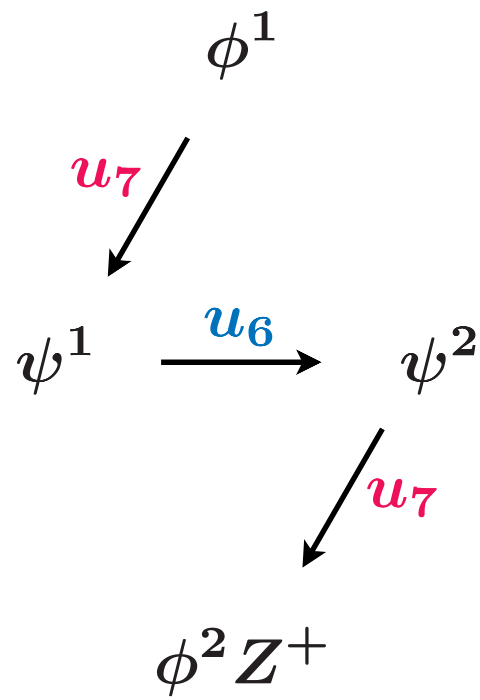
stacks (Rej-Staudacher-Zieme '07)

(Beisert-Kazakov-KS-Zarembo '05)

- k-Λ string
  - $\sin k$
  - Λ
  - $\sin k$

  $\Longleftrightarrow$  bound state of electrons

- stack
  - u_7
  - u_6
  - u_7



$$|\cdots \phi^1 \phi^1 \cdots\rangle \rightarrow |\cdots \phi^1 \phi^2 Z^+ \cdots\rangle + |\cdots \psi^1 \psi^2 \cdots\rangle$$

Correspondence of occupation numbers

$K_4 \;\Leftrightarrow\; L_H$ (length of the Hubbard model)

$K_6 \;\Leftrightarrow\; M$ (# of down spins $\downarrow$)

$K_7 \;\Leftrightarrow\; N_e$ (# of electrons $\uparrow$ & $\downarrow$)

Ground state of the Hubbard model

$N_e = L_H$ charge-singlet (half-filled)

$M = N_e/2$ spin-singlet $\uparrow\downarrow\;\uparrow\downarrow\;\uparrow\downarrow$

$\Rightarrow$

Occupation numbers for the 'AF vacuum'

$$(K_1, K_2, K_3, K_4, K_5, K_6, K_7) = (2M, M, 0, 2M, 0, M, 2M)$$

Computation of the scattering phase

- Direct computation for 2-excitations (work in progress)

- A simpler way: su(2) AF state (KS-Satoh '07)

$$K_4 = 2M = L/2, \quad L \to \infty$$

BS equations $\Longrightarrow$ effective Bethe ansatz equations

$$\left(\frac{x^+_{4,k}}{x^-_{4,k}}\right)^L \approx \prod_{j\neq k}^{K_4} \frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i} \prod_{j=1}^{M} \frac{1-g^2/x^-_{4,k}\,x^+_{2,j}}{1-g^2/x^+_{4,k}\,x^+_{2,j}} \prod_{j=1}^{M} \frac{1-g^2/x^-_{4,k}\,x^-_{2,j}}{1-g^2/x^+_{4,k}\,x^-_{2,j}}$$
$$\times \prod_{j=1}^{M} \frac{1-g^2/x^-_{4,k}\,x^+_{6,j}}{1-g^2/x^+_{4,k}\,x^+_{6,j}} \prod_{j=1}^{M} \frac{1-g^2/x^-_{4,k}\,x^-_{6,j}}{1-g^2/x^+_{4,k}\,x^-_{6,j}}$$

$$1 \approx \prod_{j\neq k}^{M} \frac{u_{6,k}-u_{6,j}+i}{u_{6,k}-u_{6,j}-i} \prod_{j=1}^{K_4} \frac{1-g^2/x^+_{6,k}\,x^+_{4,j}}{1-g^2/x^+_{6,k}\,x^-_{4,j}} \frac{1-g^2/x^-_{6,k}\,x^+_{4,j}}{1-g^2/x^-_{6,k}\,x^-_{4,j}}$$

Thermodynamic limit

$$J_0(2gt) \;=\; e^{|t|}\hat\rho_4(t) + \hat\rho_4(t) - 4g^2 t \int_0^\infty dt'\,\hat K_1(2gt, 2gt')\left[\hat\rho_2(t') + \hat\rho_6(t')\right]$$
$$0 \;=\; -e^{|t|}\hat\rho_6(t) + \hat\rho_6(t) - 4g^2 t \int_0^\infty dt'\,\hat K_0(2gt, 2gt')\hat\rho_4(t')$$

Eliminating $\hat\rho_2,\ \hat\rho_6$ $\Longrightarrow$ single integral eq.

$$J_0(2gt) = (e^{|t|}+1)\hat{\rho}_4(t) + 4g^2 t \int_0^\infty dt' \hat{K}_d(2gt, 2gt')\hat{\rho}_4(t')$$

integral eq. for
su(2) anti-ferromagnet

(Rej-Serban-Staudacher '05, Zarembo '05)

contribution
from the stacks

- Consider the integral eq. for su(2) anti-ferromagnet now in the physical description:

BS all-loop Bethe eqs. with the BES dressing factor

$$J_0(2gt) = (e^{|t|}+1)\hat{\rho}_4(t) + 4g^2 t \int_0^\infty dt' \hat{K}_d(2gt, 2gt')\hat{\rho}_4(t')$$

(Rej-Staudacher-Zieme '07)

contribution
from the dressing factor

Does this mechanism really apply to microscopic scattering?

- Small $\boldsymbol{K_4}$ has several problems

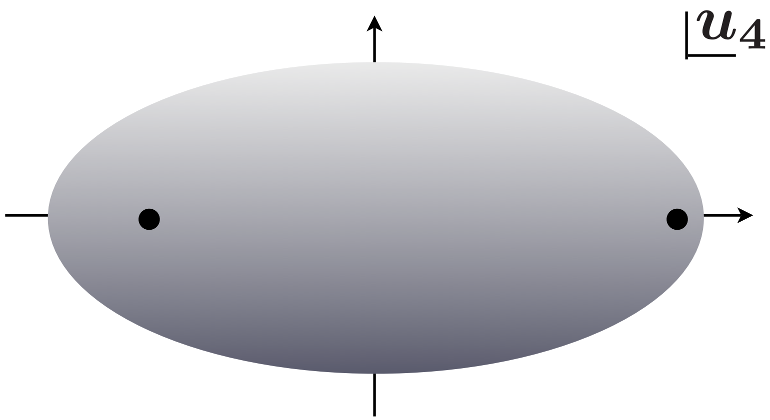
  (forbidden occupation numbers)
  (the stack solution does not exist for finite *M*)

$$\boldsymbol{K_4} \begin{cases} \boldsymbol{2M} & \Rightarrow \text{ constitute the ‘AF vacuum’} \\ + & \\ \tilde{\boldsymbol{K}}_{\boldsymbol{4}} & \Rightarrow \text{ excitations} \end{cases}$$

Need to specify the configuration of $\boldsymbol{2M}$ $\boldsymbol{u_4}$ roots

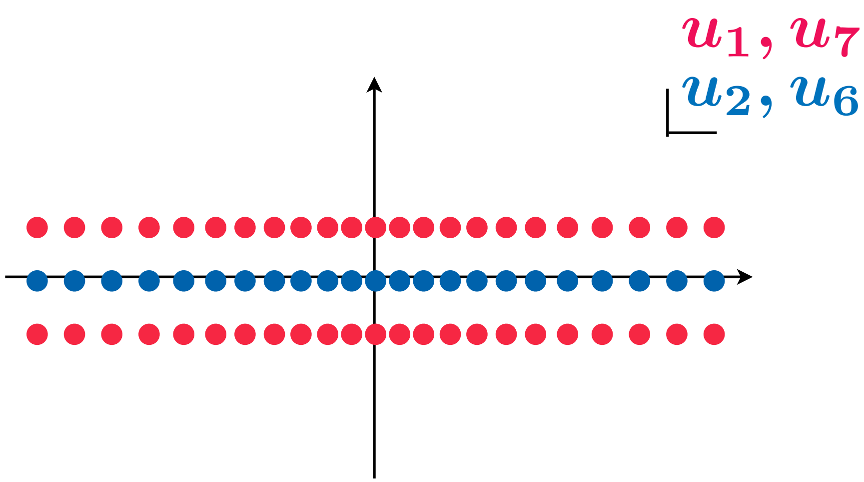
which does not affect the generation of the BES phase

(work in progress)



$$L \longrightarrow \infty$$
$$M \longrightarrow \infty$$

$2M$ u_4's + 2 excitation u_4's $\Longleftrightarrow$ (16 dim irrep.)^2



M stacks

$$\Downarrow$$

$$\mathcal{S} = S_0 \left[\mathcal{R}_{\mathfrak{su}(2|2)} \otimes \mathcal{R}_{\mathfrak{su}(2|2)} \right]$$

S-matrix with the BES factor

Conclusions

- S-matrix, including the overall scalar factor, is completely determined by the $\mathfrak{su}(2|2)$ symmetry (provided that the vacuum is uniquely singled out)

No need of gauge/string perturbative data

- Once the integrability is proven both in the Planar $\mathcal{N} = 4$ super Yang-Mills and in the free superstrings on AdS, the spectrum is uniquely constructed for arbitrary λ.

$$\Downarrow$$

Quantitative "proof" of the AdS/CFT correspondence in the limit $N \to \infty, L \to \infty$

Prospects

- Can the vacuum configuration be uniquely fixed?

- Could this approach serve as a lattice formulation?

- Deeper relation with the Hubbard model?