

# A warm-up for solving noncompact sigma models: **The Sinh-Gordon model**

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Based on

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- J.T., hep-th/0702122

# Goal: Understand string theory on AdS-spaces

Building blocks: nonlin. sigma-models on super-groups.

$$PSL(1,1|2) \leftrightarrow AdS_3 \times S^3$$

$$PSL(2,2|4) \leftrightarrow AdS_5 \times S^5$$

To write out worldsheet-actions, use Poincare coordinates:

$$ds^2 = \frac{1}{z^2} (dz^2 + dx^2).$$

Writing  $z = e^\phi \Leftrightarrow$  world-sheet Hamiltonian of the form

$$H = \dots + e^{2\phi}(\dots) + e^{-2\phi}(\dots) + \dots$$

Exponential interactions!

On the other hand: **Hope for integrability!** (Bena, Polchinski, Roiban)

## Nice recent story concerning limit (Hofman, Maldacena)

$$J \rightarrow \infty, \quad E - J = \text{fixed}, \quad \lambda \propto R_{\text{AdS}}^{-1} = \text{fixed}$$

**Gauge-fixing:**  $J \mapsto R = J/\sqrt{y}$ ,  $R$ : world-sheet radius.

$$H_{\text{g.f.}} = \int_{2\pi R}^0 d\sigma \left( \dots + e^{2\varphi}(\dots) + e^{-2\varphi}(\dots) + \dots \right)$$

**Conjecture:**  $H_{\text{g.f.}}$  is integrable.

**Proposal** for factorized scattering theory in infinite volume  $R \rightarrow \infty$  ( $\Leftrightarrow J \rightarrow \infty$ )

(Beisert, Staudacher, Hofman, Maldacena ...):

▷ Elementary excitations ("magnons")  $\mathfrak{g}_B \oplus \mathfrak{g}_F$ .

▷ Proposal for  $2 \rightarrow 2$  scattering matrix (for all  $\lambda$  iii)

**Proposal has passed highly nontrivial tests!**

# Important problem: Spektrum for finite $R \Leftrightarrow$ finite $J$ ?

**Asymptotic Bethe ansatz** (Arutyunov, Frolov, Staudacher)

- Pretend magnons are free except for magnons crossing ( $\delta$ -like interactions),

$\Rightarrow$  Ansatz for coord. space wave-fct.: Plane-waves + crossings (S-matrix)

- Periodicity of wave-fct.  $\Rightarrow$

$$e^{ip_a R} = \prod_{b \neq a} S(p_a, p_b) \cdot$$

**Problem:** Interactions are not  $\delta$ -like (vacuum polarization)

$\Rightarrow$  Expect corrections to As. Bethe ansatz!

# Classes of integrable models:

## (A) Compact integrable models.

Models associated to compact groups: WZNW,  $\sigma$ -models, XXX, XXZ, Ising, Interactions  $e^{i b \phi}$ : Sine-Gordon ...

## (B) Noncompact integrable models - Real type

Models associated to non-compact groups: WZNW,  $\sigma$ -models, XXX, XXZ, ...  
Interactions  $e^{b \phi}$ : Liouville theory, Sinh-Gordon, (Affine) Toda...

## Class (B) is important:

- String theory: Strings on noncompact target spaces (black holes, cosmology),
- Gauge theory - via AdS-CFT correspondence,
- Condensed matter: Integer quantum Hall, electron systems with disorder.

But: Class (B) is very different from (A):

— Prediction: **Bethe ansatz fails in class (B)!**

## Consider prototype: Sinh-Gordon model (on circle with circumference $R$ )

$$H_{\text{ShG}} = \int_R^0 dx \left\{ 4\pi \Pi^2 + \frac{1}{16\pi} (\partial_\sigma \varphi)^2 + 2\mu \cosh(b\varphi) \right\}.$$

### • Infinite volume $R \rightarrow \infty$ :

– Spectrum: One massive particle,  $E = m \cosh \vartheta$ ,  $p = m \sinh \vartheta$ .

– Scattering: S-matrix factorizes into two-particle scattering  $S(\vartheta_1 - \vartheta_2)$

$$S(\vartheta) = \frac{\sinh \vartheta - i \sin \vartheta_0}{\sinh \vartheta + i \sin \vartheta_0}.$$

– Fields, correlation functions: ...?!

### • Finite volume $R$ :

– Spectrum: Conserved quantities, classification of eigenvectors - today!

First example for QFT from class (B) where spectrum was understood for finite volume!

**First step:** Construct integrable lattice regularization of the Sinh-Gordon model:

- Hilbert space  $\mathcal{H}$

- Hamiltonian  $H$ ,

- set  $\mathcal{Q} = \{T_0, T_1, \dots\}$  of commuting conserved charges

## 1. Definition of $\mathcal{H}$ :

Discretize Sinh-Gordon variables as

$$\Pi^n \leftarrow \Pi(x), \quad \Delta, \quad \Phi^n \leftarrow \Phi(x), \quad \Delta, \quad x = \Delta \cdot u$$

Quantize:

$$[\Pi^n, \Phi^m] = \frac{i}{\Delta} \delta_{n,m}$$

$\Leftrightarrow$  Hilbert space  $\mathcal{H} \equiv (L^2(\mathbb{R}))^{\otimes \mathbb{N}}$ .

## 2. Construction of $\mathcal{Q}$ .

Let

$$M(n) \equiv \begin{pmatrix} A(n) & C(n) \\ B(n) & D(n) \end{pmatrix} \equiv L^N(n) \cdots L^2(n) \cdot L^1(n),$$

where

$$L(n) \equiv \begin{pmatrix} L^{11}(n) & L^{12}(n) \\ L^{21}(n) & L^{22}(n) \end{pmatrix}, \quad \begin{bmatrix} L^{11}(n) = e^{+\frac{\beta}{2}(\Pi n + 2s)} \left( 1 + e^{-\beta(\Phi n + s)} \right) \\ L^{12}(n) = \sinh(\pi \varrho n + \frac{\beta}{2} \Phi n) \\ L^{21}(n) = \sinh(\pi \varrho n - \frac{\beta}{2} \Phi n) \\ L^{22}(n) = e^{-\frac{\beta}{2}(\Pi n - 2s)} \left( 1 + e^{+\beta(\Phi n - s)} \right) \end{bmatrix} \quad \beta = \varrho \sqrt{8\pi}$$

Consider the one-parameter family of operators:

$$T(n) = \text{tr} \left( M(n) \right) = A(n) + D(n).$$

$\Leftrightarrow$  The operators  $T^m$  which are defined by  $T(n) = e^{\pi \varrho N n} \sum_{N=0}^m e^{-2\pi \varrho n} T^m$ , are positive self-adjoint and commuting,  $[T^m, T^n] = 0$ .



### 3. Construction of H

There exists an operator H which has the following properties.

(a) The operator H is local:

$$H = \sum_N^{n=1} H_{n, n+1} \cdot$$

(b) H commutes with the conserved charges  $T_k$ :

$$[H, T_k] = 0, \text{ for } k = 0, \dots, N.$$

(c) *Classical continuum limit* ( $\Pi_n \rightarrow \Pi(x), \Delta, \Phi_n \rightarrow \Phi(x), x = n\Delta$ .)

$$\left. \begin{array}{l} N \leftarrow \infty \\ \Delta \leftarrow 0 \\ s \leftarrow \infty \end{array} \right\} \text{with} \left\{ \begin{array}{l} R = N\Delta/2\pi \\ m = \frac{\Delta}{4} e^{-\pi b s} \end{array} \right\} \text{fixed.}$$

$$H \leftarrow \sum_{0 \leftarrow y}^n \frac{1}{\Delta} H_{G, cl}^{n, n+1} \leftarrow \int_{2\pi R}^0 dx \left( \frac{1}{2} \Pi^2 + \frac{1}{2} (\partial_x \Phi)^2 + \frac{\beta^2}{m^2} \cosh \beta \Phi \right) + \text{const}$$

**Good news: We do have integrable lattice regularization**

**$\Leftrightarrow$  Strategy: Diagonalize  $T(n) \Leftrightarrow \dots \Leftrightarrow$  Diagonalization of  $H$**

**Bad news: The Bethe ansatz fails:**

- one would need pseudo-vacuum  $\Omega$  annihilated by  $B(n)$
- Since interactions involve **real exponentials**  $e^{\phi_q}$ :

$B(n)\Omega = 0$  leads to equations like  $e^{\phi_q}\Omega = 0$

- **no normalizable** solution  $\Omega \in \mathcal{H}!!!$

**Reasons:**

- exponential interactions  $e^{\phi_q}$ ,
- non-compactness of target space!

However, there is something better than Bethe ansatz:

**Q-operators and separation of variables**

# Separation of Variables

Main idea (Sklyanin): Diagonalize  $B(u)$ , parameterize eigenvalues  $b(u)$  as

$$b(u) \sim \prod \sinh 2\pi b(u - y_k) \cdot$$

$\Leftrightarrow$  wave-functions  $\Psi(y_1 \dots y_N) \cdot$

Key observation (Sklyanin)  $T(u)\Psi_t = t\Psi_t(u) \Leftrightarrow$

$$t(y_k)\Psi(\dots y_k \dots) = A(y_k)\Psi(\dots y_k - ib \dots) + D(y_k)\Psi(\dots y_k + ib \dots)$$

## Eigenvalue equations separate — Baxter equation!

$\Leftrightarrow$  Seek solutions with ansatz  $\Psi_t = \prod_{k=1}^N q^t(y_k)$  where  $q^t(u)$  solution to

$$t(u)q^t(u) = A(u)q^t(u - ib) + D(u)q^t(u + ib)$$

Question: When does it give **normalizable** eigenstates of  $T(u)$ ?

Consider operator  $Q(n)$  defined on eigenstates  $\Psi^t(n)$  of  $T(n)$  by  $Q(n)\Psi^t(n) = q^t \Psi^t(n)$ ,

$$\left[ \begin{array}{l} \text{(a)} \quad Q(n)T(n) = (a(n) - i b(n))Q(n) + p(n)Q_N(n) + i q(n), \\ \text{(b)} \quad Q(n) \text{ is normal, } Q(n)Q_*(n) = Q_*(n)Q(n), \\ \text{(c)} \quad Q(n)Q(a) = Q(a)Q(n), \\ \text{(d)} \quad Q(n)T(a) = T(a)Q(n), \end{array} \right.$$

- $Q(n)$  is repackaging of conserved quantities

- Diagonalization of  $Q(n) \Leftrightarrow$  diagonalization of  $T(n)$ .

- Eigenvalues  $q(n)$  of  $Q(n)$  must satisfy the Baxter equation.

## Direct construction of $Q(n)$ (Bytsko, J.T.):

$\Leftrightarrow$  **analytic** and **asymptotic** properties of  $q(n) \Leftrightarrow$  **Quantization conditions!**

— ensure normalizability of  $\Psi^t = \prod_N^{k=1} q^t(y_k)$  !

## Full characterization of the spectrum:

A function  $t(u)$  is eigenvalue of the transfer-matrix  $T(u)$  if and only if there exists a function  $q^t(u)$  which satisfies

$$\left[ \begin{array}{l} \text{(i)} \quad t(u)q^t(u) = (a(u))_N q^t(u) - ib) + (d(u))_N q^t(u + ib), \\ \text{where } d(u) = a(-u) + 1 + \left(\frac{4}{m\Delta}\right) e^{-\pi b(2u+ib)}, \\ \text{(ii)} \quad q^t(u) \text{ is meromorphic in } \mathbb{C}, \text{ with poles in } \mp \mathcal{P}_{-s}, \\ \text{(iii)} \quad q^t(u) \sim \left. \begin{array}{l} e^{+i\pi N \sigma u - i\frac{2}{\pi} N u^2} \text{ for } |u| \rightarrow \infty, |\arg(u)| < \frac{2}{\pi} \\ e^{-i\pi N \sigma u - i\frac{2}{\pi} N u^2} \text{ for } |u| \rightarrow \infty, |\arg(u)| > \frac{2}{\pi} \end{array} \right\} \end{array} \right]$$

**Task: Classify set  $\mathcal{Q}$  of solutions to the Baxter equation (i)**

Let  $\mathcal{Y}_M$  be the set of all functions  $Y(\vartheta)$  which satisfy the integral equation

$$\log Y(\vartheta) = \int_c^{\vartheta} \frac{4\pi}{p\vartheta'} \sigma(\vartheta - \vartheta') \log(W(\vartheta') + Y(\vartheta'))$$

$$(I)_1 \quad -N \arctan \left( \frac{\sinh \sigma}{\cosh(\vartheta + i\tau)} \right) - N \arctan \left( \frac{\sinh \sigma}{\cosh(\vartheta - i\tau)} \right) + \sum_{M=1}^a \log S(\vartheta - \vartheta_a - i\frac{\tau}{2}) + \sum_{M=1}^a \log S(\vartheta - \vartheta_a - i\frac{\tau}{2}) \cdot$$

and which have the properties

$$\left[ \begin{array}{l} (Y_1)_1 \quad \log Y(\vartheta) \sim -i\delta N((\vartheta \pm \sigma \pm i\frac{\tau}{2})^2 - \tau^2) \text{ for } |u| \rightarrow \infty, |\arg(\pm u)| < \frac{\tau}{2}, \\ (Y_2)_1 \quad Y(\vartheta) \text{ is meromorphic with poles of maximal order } N \text{ in } \pm \tau - s \pm i\tau, \\ (Y_3)_1 \quad \text{There exist complex numbers } \vartheta_a \in \mathbb{S}, a = 1, \dots, M, \text{ such that} \\ W(\vartheta) + Y(\vartheta) = 0 \text{ if } \vartheta = \vartheta_a \pm i\frac{\tau}{2}. \end{array} \right]$$

$$\left( \sigma(\vartheta) \equiv \frac{4 \sin \vartheta_0 \cosh \vartheta}{\cosh 2\vartheta - \cos 2\vartheta_0}, \quad \vartheta_0 \equiv \frac{\tau b}{\pi s}, \quad \sigma \equiv \frac{\tau s}{\pi s}, \quad \tau \equiv \frac{\tau \delta'}{\pi \delta'} \right)$$

**Theorem 1.** There is a one-to-one correspondence between the solutions  $Y(\vartheta) \in \mathcal{Y}_M$  of the integral equations (I)<sub>1</sub> and the elements  $\hat{Q} \in \mathcal{Q}_M$ .

- Given  $\hat{Q} \in \mathcal{Q}_M$ , get  $Y(\vartheta)$ :

$$W(\vartheta) + Y(\vartheta) = \hat{Q}(\vartheta) + i\frac{z}{\pi} \hat{Q}(\vartheta) - i\frac{z}{\pi}.$$

$Z = \{\vartheta_1, \dots, \vartheta_M\}$ : set of zeros of  $\hat{Q}(\vartheta)$  within  $S$ .

- Given  $Y(\vartheta) \in \mathcal{Y}_M$  (zeros:  $\vartheta_a \in S \setminus \partial S$ ,  $\vartheta'_i \in \partial S$ ) get  $\hat{Q} \in \mathcal{Q}_M$  as

$$\log \hat{Q}(\vartheta) = \int_c^c \frac{d\vartheta' \log(W(\vartheta') + Y(\vartheta'))}{4\pi \cosh(\vartheta - \vartheta')} - N \arctan \left( \frac{\sinh \sigma}{\cosh \vartheta} \right)$$

$$+ \sum_{M'}^{1=a} \int_{\vartheta}^{c_a} \frac{d\vartheta' \sinh(\vartheta' - \vartheta_a)}{1} + \sum_{M-M'}^{1=b} \frac{z}{1} \int_{\vartheta}^{c_a} \frac{d\vartheta' \sinh(\vartheta' - \vartheta'_i)}{1}. \quad (I)_1$$

# Continuum limit: Easy!

Continuum limit:

$N \rightarrow \infty, s \rightarrow \infty$  such that

$$\frac{mR}{b} \frac{2 \sin \pi \frac{2\delta}{b}}{2N \exp\left(-\frac{\pi}{2\delta} s\right)} \equiv$$

is kept constant.

$\Leftarrow \dots \Leftarrow \dots \Leftarrow$

Main claim:



## Main claim:

The Hilbert space of the Sinh-Gordon model contains (is equal to?)  $\mathcal{H}^{\text{TBA}} = \bigoplus_{M=0}^{\infty} \mathcal{H}^M$ . The spaces  $\mathcal{H}^M$  have ONB  $e_k$  labelled by  $\mathbf{k} = (k_1, \dots, k_M) \in \mathbb{Z}^M$ ,  $k_1 > \dots > k_M, e_k$ : eigenvect. to the conserved quantities of the model.

The corresponding function  $\mathcal{Q}^{\mathbf{k}}(\vartheta)$  can be represented as

$$(C) \quad \log \mathcal{Q}^{\mathbf{k}}(\vartheta) = \int_{\mathbb{R}} \frac{d\vartheta'}{2\pi} \log(W(\vartheta') + Y^{\mathbb{T}^{\mathbf{k}}}(\vartheta')) - mR \frac{2 \sin \vartheta_0}{\cosh \vartheta} + \sum_M^{a=1} \int_{c_a}^{\vartheta} d\vartheta' \frac{\sinh(\vartheta' - \vartheta^a)}{1}$$

where  $\mathbb{T}^{\mathbf{k}} = [\vartheta_1, \dots, \vartheta_M]$  is the unique solution of

$$(B) \quad 2\pi k_a + mR \sinh \vartheta^a + \sum_M^{b=1, b \neq a} \arg S(\vartheta^a - \vartheta^b) + i \int_{\mathbb{R}} \frac{d\vartheta}{2\pi} \sigma(\vartheta^a - \vartheta) + i \frac{z}{2} \log(1 + Y^{\mathbb{T}^{\mathbf{k}}}(\vartheta)) = 0,$$

with  $Y^{\mathbb{T}^{\mathbf{k}}}(u)$  being defined as the unique solution to

$$(A) \quad \int_{\mathbb{R}} \frac{d\vartheta'}{2\pi} \sigma(\vartheta - \vartheta') \log(1 + Y^{\mathbb{T}^{\mathbf{k}}}(\vartheta')) + mR \cosh \vartheta + \sum_M^{a=1} \log S(\vartheta - \vartheta^a) - i \frac{z}{2} = 0.$$

The energies are calculated from  $Y_{\mathbb{T}}(\vartheta)$  as follows:

$$E_k = \sum_{a=1}^M m \cosh \vartheta_a - m \int_{\mathbb{R}} \frac{d\vartheta}{2\pi} \cosh \vartheta \log(1 + Y_{\mathbb{T}}(\vartheta))$$

**Excited state TBA !**

(Generalizes work of A.I.B. Zamolodchikov, S. Lukyanov for the ground state)

## The IR limit $R \rightarrow \infty$

Considering IR limit  $R \rightarrow \infty$ , notice that  $Y_{\mathbb{T}}(\vartheta) = \mathcal{O}(e^{-mR})$ .

$\Rightarrow$

### Particle picture:

- $M$ : number of particles,
- $\vartheta_a$ : Rapidity of particle  $a$ ,

$$E_a = m \cosh \vartheta_a, \quad p_a = m \sinh \vartheta_a,$$

- $\vartheta_a$  quantized by "asymptotic Bethe equations".

$$e^{-iR p_a} \prod_{\substack{b=1 \\ b \neq a}}^M S(\vartheta_a - \vartheta_b) = 1$$

**For finite  $R$ :**

$$e^{-iR(p_a + \Phi_a)} = \prod_{\substack{b=1 \\ b \neq a}}^M S(\vartheta_a - \vartheta_b),$$

where

$$\Phi_a \equiv \int_{\mathbb{R}} \frac{d\vartheta}{2\pi R} \sigma(\vartheta_a - \vartheta + i\frac{z}{\pi}) \log(1 + Y_{\mathbb{T}}(\vartheta)) = \mathcal{O}(e^{-mR}).$$

**$\Phi_a$ : Effects of vacuum polarization!**

**The UV limit  $R \rightarrow 0$ :**

**— Connection with Liouville theory!**

## Compare with “String Bethe Ansatz”:

- Valid for  $J \rightarrow \infty$ ,  $J$ : angular momentum on  $S^5$   
~ Radius of world-sheet cylinder in gauge-fixed action.
- For finite  $J$ : corrections  $\mathcal{O}(e^{-2\pi J/\sqrt{\lambda}})$  to String Bethe Ansatz  
(Schäfer–Nameki, Zamaklar, Zarembo)  
≡ counterparts of  $e^{-m_H}$ -corrections in Sinh-Gordon
- Gauge theory side: “Wrapping interactions”:  
(see Kotikov, Lipatov, Rej, Staudacher, Velizhanin)

## Summary

- Solved first example of noncompact integrable QFT model: Sinh-Gordon

- Bethe ansatz fails but Separation of Variables works.

- Remarkable self-duality  $b \leftrightarrow b^{-1}$  —

— variant of Langlands-duality....

- Prediction: Bethe ansatz fails for  $AdS_5$  sigma model, finite  $J$ .

But don't forget:

There is sthg. better than Bethe ansatz!