INTRODUCTION TO SEMICLASSICAL ANALYSIS

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1. INTRODUCTION TO THE COURSE

1.1. What is this all about? Objective: analyse qualitatively and quantitatively certain types of linear differential operators appearing in mathematical physics.

• Quantum mechanics of a massive, nonrelativistic particle is governed by the time-dependent Schrödinger eq. on Hilbert space \( \psi(t) \in L^2(\mathbb{R}^d) \),

\[
\tag{1.1}
 i\hbar \partial_t \psi(x,t) = \left( -\frac{\hbar^2 \Delta}{2m} + V(x) \right) \psi(x,t)
\]

The differential operator \( P_h = -\frac{\hbar^2 \Delta}{2m} + V(x) \) is called the quantum Hamiltonian of the system. Here \( \hbar \approx 10^{-34} \text{J.s} \) has the dimension of an action, \( m \) is the mass of the particle (ex: \( m_{\text{electron}} \approx 10^{-19} \text{kg} \)), \( V(x) \) is a scalar potential (e.g. electric).

• secondary application (won’t be treated here): wave equation on \( \mathbb{R}^d \) or on a riemannian manifold \( X \)

\[
(\partial_t^2 - c^2 \Delta) u(x,t) = 0
\]

We will reduce the dimension of the equation by imposing a mass \( m \overset{\text{def}}{=} 1 \). We will also fix a reference energy scale \( E_0 \) typical of the values of \( V \), and set it to unity, so there remains only a single dimension (length=time), and we are lead to study the operator

\[
\tag{1.2}
P_h = -\frac{\hbar^2 \Delta}{2} + V(x)
\]

around the energy \( \sim 1 \) (adimensional). We notice that now \( [h] = \text{length} = \text{time} \), the unique remaining dimension. We can then consider the typical length scale \( L_0 \) set by \( V(x) \) (around the energy \( \sim 1 \)), and remove the last dimension by setting \( L_0 = 1 \). All quantities are now dimensionless, and \( \hbar \) takes a certain numerical value. Is it small? Large? Medium? Notice that the original \( \hbar \) had the dimension of an action \([\hbar] = [ET]\), so given the 3 dimensional quantities \( E_0, m, L_0 \), we see that the only action we can construct is \( S_0 = E_0^{1/2}m^{1/2}L_0 \). Physically, we should thus compare \( \hbar = 10^{-34}SI \) to the value of \( S_0 \).

The operators involved will be self-adjoint on \( L^2(\mathbb{R}^d) \), and therefore admit real spectra. One major goal will be to analyze quantitatively the spectra of such operators. In the cases we will study, spectra will often have a discrete component made of isolated eigenvalues of isolated multiplicities

\[
\tag{1.3}
P_h \psi_i = E_i \psi_i
\]

In this situation we will be interested in

(1) the distribution of the eigenvalues \( \{ E_{i,h} \} \), in some interval \( \sim 1 \).
(2) the spatial properties of the eigenfunctions $\psi_{i,h}$.

The discrete spectrum is made of isolated eigenvalues $E_i$ of finite multiplicities. In the generic case, these eigenvalues are simple, associated with eigenstates (eigenfunctions) $\psi_i \in \mathcal{D}(P)$.

1.1.1. Semiclassical limit = fast oscillatory functions. As often the case in analysis, one can compute effectively only in presence of a small parameter, meaning in some asymptotic limit. The semiclassical limit consists in analyzing the operator $P_\hbar$ in the regime $\hbar \ll 1$. This corresponds to wavefunctions $u(x,t)$ which oscillate fast both in time and position, compared with the global scales ($\sim 1$).

A local model to keep in mind is that of the pure Laplacian ($V \equiv 0$ locally). Then, we find that the local equation $-\hbar^2 \Delta u = 1$ can be solved locally by linear combinations of plane waves with same wavelength $2\pi \hbar \ll 1$:

$$u(x) = \int_{S^{d-1}} c(\xi) e^{i\xi \cdot x/\hbar} d\xi.$$ 

These waves oscillate on scales $\sim \hbar$ much smaller than the global scales of the problem ($L_0 \sim 1$).

We note that such oscillatory functions can be very complicated (cf. pictures of eigenmodes of billiards). Any kind of analysis would naively seem hopeless. Also, because $\hbar$ is in front of the most singular term (in the PDE sense: the highest derivative term), the limit $\hbar \to 0$ of the Schrödinger equation is certainly very singular.

What do we gain from studying this semiclassical regime?

Claim. The semiclassical regime allows us to relate the Schrödinger equation (a linear PDE) with the classical mechanics of point particles (a nonlinear ODE).

1.2. Quantum Mechanics in a nutshell.

1.2.1. Wavefunctions and probability distributions. Quantum Mechanics was developed, as a (pretty strong) modification of classical mechanics, more precisely Hamilton’s formulation of conservative (dissipationless) classical mechanics, which we will review in section 2 below.

In classical mechanics, the state of a particle at time $t$ is uniquely and precisely described by the data of its position $x(t)$ and its velocity $\dot{x}(t)$, or equivalently its momentum (“impulsion”) $\xi(t)$. Mathematically, a difference between the two points of view is that $\dot{x}(t) \in T\mathbb{R}^d$ is a tangent vector, while $\xi(t) \in T^*\mathbb{R}^d$ is a cotangent vector. This difference is not really relevant when working on $\mathbb{R}^d$. In quantum mechanics, the state of a particle is much more involved to describe, and much richer. The state of a particle (say, an electron) is given by a wavefunction $\psi(x,t)$, which is a time-dependent, complex-valued function $\psi(t) \in L^2(\mathbb{R}^d)$ with unit norm. Usually $\psi(t)$ also enjoys some further localization and regularity property, so as to lie in the domain of some differential operator.

What is the meaning of the wavefunction $\psi(t)$? It represents the particle as a wave, which is then intrinsically delocalized. Alternatively, it describes a point particle, the position/momentum of which cannot be known deterministically, but only probabilistically. That is, if one performs a position measurement on the particle at time $t$, one cannot in advance predict the outcome of the measurement, but only provide a probability distribution of the outcome. Quantum mechanics is intrinsically a probabilistic theory.
The probability distribution is given by the function $|\psi(t, x)|^2$. This is indeed a probability distribution, from the assumption that $\psi(t)$ is $L^2$-normalized. The function $\psi(x, t)$ is often called the position probability amplitude.

What is the use for the complex phase of $\psi(t, x)$? To determine the probability distribution of the momentum of the particle. If one proceeds with a momentum measurement (what people actually do in particle accelerators), one cannot predict the outcome, but only the probability distribution of this outcome. The probability density is given by $|\hat{\psi}(\xi, t)|^2$, where

$$\hat{\psi}(\xi, t) \equiv F_h \psi(\xi, t) \equiv \frac{1}{(2\pi\hbar)^{d/2}} \int dx \ e^{-i\xi \cdot x/\hbar} \psi(x, t)$$

represents the momentum probability amplitude. From Parseval’s identity, $\hat{\psi}$ is also $L^2$-normalized, so that $|\hat{\psi}|^2$ is indeed a probability density.

As a result, the same function $\psi$ allows to represent both the position and momentum probability distributions. Clearly, changing the phase of $\psi(x, t)$ won’t change $|\psi(t, x)|^2$, but will influence quite drastically the density $|\hat{\psi}(\xi, t)|^2$.

1.2.2. Observables in classical and quantum mechanics. If the wavefunction $\psi(x)$ is nice enough, say if $\psi \in S(\mathbb{R}^d)$, then distributions of the position or momentum variables can be described through their moments. Namely, for any multi-index $\alpha \in \mathbb{N}^d$, the moment of this variable, $E_x x^\alpha$, is finite. This moment can be interpreted as a “diagonal matrix element” of a corresponding multiplication operator

$$\text{Op}(x^\alpha) : \psi(x) \mapsto x^\alpha \psi(x).$$

For $\alpha \neq 0$ this operator is unbounded on $L^2$, but it has a dense domain on which it is selfadjoint. The function $\psi \in S$ obviously lies in the domain of this operator, and we have

$$E_x x^\alpha = \int dx |\psi(x)|^2 x^\alpha = \langle \psi, \text{Op}(x^\alpha) \psi \rangle.$$

Similarly, the moments of the momentum variables can be viewed as matrix elements of corresponding momentum operators. Indeed, for any multi-index $\alpha$ the momentum of the momentum variable $\xi^\alpha$, for a particle in the state $\psi(x)$, is defined as

$$E_x \xi^\alpha = \int d\xi |\hat{\psi}(\xi)|^2 \xi^\alpha = \langle \hat{\psi}, \xi^\alpha \hat{\psi} \rangle_{L^2(\xi)}.$$

Now, we would like to express this matrix element in terms of the original wavefunction $\psi(x)$. A straightforward computation shows that the multiplication operator by $\xi^\alpha$ is transformed, through the $\hbar$-Fourier transform, into the differential operator $\left(\frac{\p}{\p t}\right)^\alpha$:

$$\left[ F^{-1}_h \left( \xi^\alpha \hat{\psi}(\xi) \right) \right](x) = \left[ \left( \frac{\hbar}{i} \frac{\partial}{\partial t} \right)^\alpha \right] \psi(x).$$

---

1. $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ with $\alpha_i \in \mathbb{N}$, and we note $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$. Similarly, we will note the multi-derivative $\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$, with $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

2. In physics such expectation values (“averages”) are usually denoted with brackets, $E_x x^\alpha = \langle x^\alpha \rangle$. We will try not to use this notation, to avoid any confusion with scalar products.
Since $\psi$ is obviously in the domain of the operator $\text{Op}_h(\xi^\alpha) \overset{\text{def}}{=} (\frac{\hbar}{i})^\alpha$, we may write
\[
E_\psi \xi^\alpha = \langle \psi, \text{Op}_h(\xi^\alpha) \psi \rangle,
\]
where the scalar product is in $L^2(x)$.

Here we have let correspond:

- to the position variable $x_j$ the operator of multiplication by $x_j$, which we denote $\text{Op}(x_j) = \text{Op}_h(x_j)$ (notice that this operator is independent of $\hbar$).

- to the momentum variable $\xi_j$ the differential operator $\frac{\hbar}{i} \frac{\partial}{\partial x_j} \overset{\text{def}}{=} \text{Op}_h(\xi_j)$. (this operator explicitly depends on $\hbar$).

The Fourier transform should be interpreted as a “change of basis” on the Hilbert space of states, which exchanges a multiplication operator with a differential operator.

Position and momentum form a first set of (important) variables describing our particle in phase space. Such variables, which can be experimentally measured, are called observables in quantum mechanics. More generally, one calls classical observable a smooth, real valued function on phase space $a \in C^\infty(\mathbb{R}^{2d}, \mathbb{R})$, while a quantum observable is a selfadjoint operator $A$ on $L^2$ (sometimes unbounded, in which case we’ll assume that they have a dense domain in $L^2$). A classical observable can be used to test the position of a classical particle, or of a distribution $\rho(x,\xi)$ of particles in phase space through classical averages $\int \rho(x,\xi)a(x,\xi)dx \, d\xi$. Similarly, quantum observables can be seen as “test operators”, helping to grab the structure of the wavefunction $\psi$ through quantum averages $\langle \psi, A \psi \rangle$.

**Remark 1.1.** What is the interpretation of such a quantum average? As a selfadjoint operator, the operator $A$ can be (at least in a though experiment) experimentally measured. For a general state $\psi$, the outcome of the measurement cannot be predicted with certainty, but is described by a random variable, which will change if we repeat the measurement many times (re-constructing the same wavefunction $\psi$ before each measurement). Assuming $\psi$ is nice enough, that random variable can be described in terms of its moments, which are given by the matrix elements $\langle \psi, A^n \psi \rangle$, $n \geq 1$.

Quantum mechanics establishes a correspondence between both sets of observables, through a quantization procedure
\[
a \in C^\infty(\mathbb{R}^{2d}) \mapsto A = \text{Op}_h(a)
\]
mapping a classical observable (a function on phase space) into a quantum observable (an operator on $L^2$). This quantization procedure lies at the heart of semiclassical analysis. Most of our lectures will be devoted to a precise study of this quantization procedure and of its consequences, like the quantum-classical correspondence.

We won’t give yet the precise definition of this quantization, but only a few relevant properties:

1. the position monomials $x^\alpha$ are quantized into the corresponding multiplication operators
2. the momentum monomials $\xi^\alpha$ are quantized into the above differential operators
3. quantization is a linear operation,
4. quantization should map real valued functions into (essentially) selfadjoint operators.
The last property is specific to quantum mechanics, and will lead to the so-called Weyl quantization. In the study of linear PDEs is is often customary to introduce different (yet related) forms of quantization, which do not necessarily satisfy requirement 4.

By taking linear combination of monomials we get polynomials, which are in some sense “dense” in the set of smooth functions. Hence, it is reasonable to expect the following extension of the above properties:

1’. a smooth function (which does not grow too wildly at infinity) \( a(x) \) is quantized into the multiplication by \( a(x) \).

2’. a smooth function \( b(\xi) \) is quantized into the pseudodifferential operator \( b \left( \frac{\hbar}{i} \partial \right) \).

(Of course we’ll need to be more precise on the growth conditions we have to impose. We will be lead to introduce functional spaces of symbols (=\( \hbar \)-dependent smooth functions) with appropriate growth properties.

**Problem 1.2.** The natural question to ask is: how does one quantize a function (say, a polynomial) depending on both \( x \) and \( \xi \)? From the above properties we naturally come into ordering questions.

Indeed, the novelty in QM, is that operators \( \text{Op}_\hbar(x_j) \) and \( \text{Op}_\hbar(\xi_j) \) do not commute, but satisfy the commutation relations

\[
[\text{Op}_\hbar(x_j), \text{Op}_\hbar(\xi_k)] = i\hbar \delta_{jk} \quad \text{(where } \delta \text{is the Kronecker symbol)}.
\]

What should then be the quantization of the observable \( x_j \xi_j \)? We easily check that neither \( A_1 \overset{\text{def}}{=} \text{Op}_\hbar(x_j) \text{Op}_\hbar(\xi_j) \) nor \( A_2 \overset{\text{def}}{=} \text{Op}_\hbar(\xi_j) \text{Op}_\hbar(x_j) \) are symmetric operators, so they don’t satisfy the requirement 4. We’ll see that it suffices to take the average of the two, namely \( \frac{A_1 + A_2}{2} \), to “do the job”.

Such ordering problems, coming from the noncommutation of operators, are also at the heart of the Heisenberg uncertainty principle TD 1.

1.2.3. *Time evolution in quantum mechanics.* From this interpretation of \( \psi(t) \), one is naturally interested in the time evolution of the wavefunction to understand how the particle evolves. In quantum mechanics, and in absence of magnetic fields, this evolution is governed by the Schrödinger equation (1.1), where \( V(x) \) is the potential energy of the particle at point \( x \).

The self-adjointness of the operator \( P_\hbar \) allows to prove (using the spectral theorem and the associated functional calculus) that for any intial data \( \psi_0 \) in the domain of \( P_\hbar \), the Schrödinger equation is globally well-posed, and admits a solution \( \psi(t) \) for all \( t \in \mathbb{R} \). The solution \( \psi(t) \) is unique, and its \( L^2 \)-norm is preserved by the evolution. Actually, functional calculus allows to define the propagator

\[
U(t) \overset{\text{def}}{=} \exp \left( -itP_\hbar/\hbar \right),
\]

which forms a group of *unitary* operator on \( L^2 \rightarrow L^2 \), so that the state

\[
(1.5) \quad \psi(t) = U(t)\psi_0
\]
solves the Schrödinger equation (1.1). Notice that the propagator is bounded on \( L^2 \), so that (1.5) makes sense even when \( \psi_0 \) is not in the domain of \( P_\hbar \).
Remark 1.3. This unitarity is interpreted by the fact that the particle does not lose mass (=probability) in the evolution: our particles do not disintegrate or lose mass by emitting energy. To treat such phenomena (which are very relevant in high energy physics) one has to enlarge the quantum theory: the Schrödinger equation does not suffice any more, and one enters the (fascinating) realm of Quantum Field Theory.

Most actual particles (like the electron, photon, proton...) carry an intrinsic magnetic momentum, called spin, which forces to describe them not as scalar functions $\psi(x, t)$, but as vector valued functions $\vec{\psi}(x, t)$, where the vector (actually called a spinor) has dimension 2 or more, depending on the type of the particle. The state space is thus rather $L^2(\mathbb{R}^d, \mathbb{C}^2)$. To simplify our presentation we will exclusively deal with scalar particles, namely particles without spin degrees of freedom.

2. Hamilton’s formulation of classical mechanics

2.1. From Newton to Hamilton. A classical particle on $\mathbb{R}^d$ is described by a trajectory $x(t) \in \mathbb{R}^d$. At each time $t$ it occupies a single point $x(t) \in \mathbb{R}^d$, and has a velocity $\dot{x}(t) = \frac{dx(t)}{dt} \in \mathbb{R}^d$. The motion is determined by Newton’s law (1st principle of mechanics):

$$m\ddot{x}(t) = F(x(t)), \tag{2.1}$$

where $F : x \mapsto F(x) \in \mathbb{R}^d$ is the force field at position $x$ (here we assume this force field to be time independent). Since this equation is of second order in time, ODE theory shows that, provided $F(x)$ is smooth near $x(0)$, the initial data $(x(0), \dot{x}(0))$ suffice to specify, at least locally in time, the trajectory $(x(t))_{t \in I}$.

Remark 2.1. The trajectory may have singularities at finite time, e.g. one may have $x(t) \xrightarrow{t \to T} \infty$ even if $F$ is smooth everywhere. This cannot be the case under appropriate conditions on the force field $F$ (smooth enough, and does not grow too fast at infinity).

In the following we will always assume that the trajectories remain finite for any real time, so that $(x(t))_{t}$ is well-defined for all $t \in \mathbb{R}$. One then says that the flow is complete.

The force field $F(x)$ is said to be conservative if it derives from a potential energy (which we will call “potential” for short) $V : x \mapsto V(x) \in \mathbb{R}$:

$$F(x) = -\nabla V(x). \tag{2.2}$$

In this case, the the total mechanical energy

$$E(x, \dot{x}) = E_{\text{kin}} + V = \frac{m|\dot{x}|^2}{2} + V(x)$$

is preserved during the evolution: $\frac{d}{dt} E(x(t), \dot{x}(t)) = 0$.

\[^3\] the negative sign implies that the particle “rolls down” the energy landscape: it is attracted by low values of the potential.
It is customary (and actually far-reaching) to slightly change variables, by defining the momentum of the particle, which in this Euclidean setting reads\(^4\)
\[
\xi(t) = m\dot{x}(t).
\]
The dynamical variables specifying the motion of the particle are now \((x(t), \xi(t)) \in \mathbb{R}^d \times \mathbb{R}^d\). The mechanical energy can now be cast into Hamilton’s function
\[
H(x, \xi) = \frac{|\xi|^2}{2m} + V(x),
\]
a function over the phase space \(\mathbb{R}^{2d} = \mathbb{R}^d_x \times \mathbb{R}^d_\xi\). We will sometimes denote by \(\rho = (x, \xi)\) a phase space point.

After this change of variables, Newton’s law (2d order eq. on \(d\) variables) can be cast into Hamilton’s equations over the phase space (1st order eqs. on \(2d\) variables):
\[
\begin{aligned}
\dot{x}(t) &= \frac{\partial H}{\partial \xi}(x(t), \xi(t)) \\
\dot{\xi}(t) &= -\frac{\partial H}{\partial x}(x(t), \xi(t))
\end{aligned}
\iff \dot{\rho}(t) = X_H(\rho)
\]

The RHS defines the Hamiltonian vector field \(\rho \in \mathbb{R}^{2d} \mapsto X_H(\rho) \in T_\rho \mathbb{R}^{2d} \equiv \mathbb{R}^{2d}\), which generates the Hamiltonian flow associated with the hamilton function \(H\):
\[
\Phi^t_H : \rho(0) = (x(0), \xi(0)) \in \mathbb{R}^{2d} \mapsto \Phi^t_H(\rho(0)) = \rho(t) = (x(t), \xi(t)) \in \mathbb{R}^{2d}.
\]

Being a flow means that (provided everything is well-defined),
\[
\Phi^{t+s}(\rho) = \Phi^t(\Phi^s(\rho)),
\]
both for positive or negative times.

Remark 2.2. This Hamiltonian formalism is not restricted to functions of the form (2.3), but can be generalized to arbitrary (smooth enough) functions \(H(x, \xi)\) on phase space. Most of what we will say in this subsection applies in this higher generality, and defines a Hamiltonian flow on \(\mathbb{R}^{2d}\).

Here again, the flow may not be defined for all time.

Remark 2.3. We will assume that \(H \in C^\infty(\mathbb{R}^{2d})\), and that the flow \(\Phi^t_H\) is complete. In tutorial 2, we give a simple example of incomplete flow, in \(d - 1\).

In this context, the conservation of energy now reads as follows:

Proposition 2.4. The Hamiltonian flow \(\Phi^t_H\) leaves invariant the value of the Hamiltonian.
\[
\forall \rho, \forall t \in \mathbb{R}, \ H(\Phi^t_H(\rho)) = H(\rho)
\]

Proof. Explicit computation using Hamilton’s equations (2.4)
\[
\frac{dH}{dt}(x, \xi) = \frac{\partial H}{\partial x} \dot{x} + \frac{\partial H}{\partial \xi} \dot{\xi} = 0.
\]

\(^4\)Beware that we will use PDE’s notation \((x, \xi)\) for the position-momentum. In classical mechanics and quantum mechanics, one rather uses the notations \((x, p)\), or also \((q, p)\), like in symplectic geometry. Similarly, the notation \(\rho = (x, \xi)\) for a phase space point seems typical of PDEs; symplectic geometers prefer \(x = (q, p)\)!
As a consequence, for any $H$ the phase space $\mathbb{R}^{2d}$ is \textbf{foliated into energy layers} (or hypersurfaces)

$$\Sigma_E \overset{\text{def}}{=} H^{-1}(E) = \{ \rho \in \mathbb{R}^{2d}, H(\rho) = E \},$$

and each layer $\Sigma_E$ is invariant through the flow $\Phi^t_H$. Hence, one can study the property of the flow $\Phi^t_H$ restricted to a single energy layer.

**Definition 2.5.** A fixed point for the flow $\Phi^t_H$ is a point $\rho_c \in \mathbb{R}^{2d}$ for which $X_H(\rho_c) = 0$. Such a point is called \textbf{critical}. The corresponding energy $H(\rho_c)$ is called a critical energy.

The implicit function Thm shows that, assuming that if the energy $E$ is noncritical, then $\Sigma_E$ is a smooth embedded hypersurface.

### 2.2. Symplectic structure.

This formulation of conservative mechanics is equivalent with Newton’s formulation. What we gain is a explicit new invariant structure, namely the \textbf{symplectic structure} on $\mathbb{R}^d \times \mathbb{R}^d$, which is explicitly given by the nondegenerate 2-form on that vector space, which we denote by\(^5\)

$$\omega = \sum_{j=1}^d d\xi_j \wedge dx_j. \tag{2.5}$$

This notation means that for any two vectors $V = (V_\xi, W_\xi) \in \mathbb{R}^{2d}$,

$$\omega(V, W) = \sum_j V_\xi W_x - V_x W_\xi = (JV, W),$$

where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

For any point $\rho \in \mathbb{R}^{2d}$, we may apply this formula to any pair of tangent vectors $V, W \in T_\rho \mathbb{R}^{2d}$, which defines $\omega$ as a differential 2-form on $\mathbb{R}^{2d}$. This 2-form is obviously \textbf{nondegenerate} and \textbf{closed}.

The Hamiltonian vector field can be defined using the symplectic form, by the following compact equation:

$$\iota_{X_H} \omega = -dH \iff \forall V \in T_\rho \mathbb{R}^{2d}, \quad \omega(X_H, V) = -dH(V) \tag{2.6}$$

**Definition 2.6.** The symplectic form generates the \textbf{Poisson bracket} on $\mathbb{R}^{2d}$. For any pair of functions $f, g \in C^1(\mathbb{R}^{2d})$, the bracket is defined by

$$\{f, g\} \overset{\text{def}}{=} \sum_{j=1}^d \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial \xi_j} = X_f g = -X_g f = \omega(X_f, X_h)$$

If $f = H$ is the Hamiltonian, we have

$$\{H, g\} = X_H g = -dH(X_g),$$

\(^5\)Most physics or mechanics books choose instead $\omega = \sum_{j=1}^d dx_j \wedge d\xi_j$, but this sign change is just a matter of convention. We will use the present convention to conform with PDE notations.
defines the infinitesimal change of the observable $g$ evolved along the Hamiltonian flow $X_H$:

\[ \{H,g\} = \frac{d}{dt} g \circ \Phi^t_H \mid_{t=0}. \]

**Proposition 2.7.** The Hamiltonian flow $\Phi^t_H$ preserves the symplectic form. In other words, the pull-back of $\omega$ through the flow is equal to $\omega$:

\[(\Phi^t_H)^* \omega = \omega\]

**Proof.** One can write down the explicit infinitesimal transformation of $\omega$ under the vector field $X_H$, in a slightly sloppy way:

\[
\dot{\omega} = \sum d\xi_j \wedge dx_j + d\xi_j \wedge d\dot{x}_j \\
= \sum -d (\partial_{x_j} H) \wedge dx_j + d\xi_j \wedge d (\partial_{\xi_j} H) \\
= \sum - (\partial_{x_j x_k} H dx_k + \partial_{x_k} H dx_j + \partial_{\xi_j \xi_k} H d\xi_k) \wedge dx_j + d\xi_j \wedge \left( \partial_{\xi_j x_k} H dx_k + \partial_{\xi_j} H d\xi_k \right)
\]

The cross-terms $d\xi_k \wedge dx_j$ cancel each other. We may now invoke the fact that $(\partial^2_{x_j x_k} H)$ and $(\partial^2_{\xi_j \xi_k} H)$ are symmetric matrices, while $dx_k \wedge dx_j$ and $d\xi_j \wedge d\xi_k$ are antisymmetric, to kill the remaining terms, and get $\dot{\omega} = 0$.

A faster (and more geometric proof) uses the Cartan formula:

\[ \dot{\omega} = \mathcal{L}_{X_H} \omega = d (\iota_{X_H} \omega) + \iota_{X_H} d\omega. \]

The closedness of $\omega$ kills the second term. On the other hand, $\iota_{X_H} \omega = -dH$, so we get zero. \qed

2.2.1. **Lagrangian subspaces and Lagrangian submanifolds.** The symplectic form $\omega$ allows to split the subspaces of $\mathbb{R}^{2d}$ into various classes, according to their properties w.r.t. $\omega$.

**Definition 2.8.** For any subspace $W \subset \mathbb{R}^{2d}$, its symplectic orthogonal, or symplectic complement, is the subspace defined by $W^\perp \overset{\text{def}}{=} \{V \in \mathbb{R}^{2d}, \forall V' \in W, \omega(V,V') = 0\}$.

Due to the non-degeneracy of $\omega$, one always has $\dim W^\perp = 2d - \dim W$, and $(W^\perp)^\perp = W$.

This form of orthogonality enjoys quite different properties from the usual orthogonality (w.r.t. a scalar product).

**Definition 2.9.** A subspace $W \subset \mathbb{R}^{2d}$ is called isotropic if $W \subset W^\perp$, or equivalently $\omega \mid_W = 0$. This means that for any vectors $V,W \in I$, one has $\omega(V,W) = 0$.

A subspace $W \subset \mathbb{R}^{2d}$ is called co-isotropic if $W^\perp \subset W$.

A subspace $W \subset \mathbb{R}^{2d}$ is called lagrangian iff $W = W^\perp$. Equivalently, $W$ is both isotropic and co-isotropic. Equivalently, $W$ is isotropic of maximum dimension, that is dimension $d$.

This linear description can be extended to smooth submanifolds on $\mathbb{R}^{2d}$. A smooth submanifold $\Lambda \subset \mathbb{R}^{2d}$ is said to be isotropic/coisotropic/lagrangian if, at each point $\rho \in X$, the space $T_{\rho} \Lambda \subset T_{\rho}\mathbb{R}^{2d} \simeq \mathbb{R}^{2d}$ is isotropic/coisotropic/lagrangian.
2.2.2. Generating function of a Lagrangian subspace. One way to characterize a Lagrangian subspace, is that its graph (in the appropriate set of coordinates) can be expressed through the gradient of a scalar function, named the generating function of that Lagrangian submanifold.

Let us start with a Lagrangian subspace \( W \subset \mathbb{R}^{2d} \). Since the subspace \( W \) has dimension \( d \), we want to see it as the graph of a function on \( d \) coordinates. For instance, if the above matrix \( M \) is invertible, we could also write \( W = \{ (x, \xi = Mx), \; x \in \mathbb{R}^d \} \), where \( M \) is a \( d \times d \) real matrix. It is easy to check that such a subspace is Lagrangian if and only if \( M \) is symmetric. In that case, we notice that \( Mx \) can be obtained as the gradient of the quadratic form \( Q_M(x) = \frac{1}{2} \langle x, Mx \rangle \), therefore

\[
W = \left\{ (x, \xi = \nabla Q_M(x)), \; x \in \mathbb{R}^d \right\}.
\]

The quadratic form \( Q_M(x) \) is a generating function for \( W \).

A generic subspace can be projected on several sets of coordinates. For instance, if the above matrix \( M \) is invertible, we could also write \( W = \{ (x = M^{-1} \xi, \xi), \; \xi \in \mathbb{R}^d \} \). The matrix \( M^{-1} \) is automatically symmetric, so here as well it makes sense to use a generating function. For reasons explained below, we will choose the quadratic form with opposite sign, \( \tilde{Q}(\xi) = -\frac{1}{2} \langle \xi, M^{-1} \xi \rangle \) to generate \( W \) in the \( \xi \) coordinates. There is a systematic procedure to recover \( \tilde{Q}(\xi) \) from the knowledge of \( Q(x) \), it is the Legendre transform. Namely, given \( \xi \in \mathbb{R}^d \), consider the function

\[
(2.8) \quad F_\xi(x) = Q(x) - \langle x, \xi \rangle.
\]

It is easy to check that this function admits a unique critical point:

\[
\nabla_x F_\xi(x) = 0 \iff Mx - \xi = 0 \iff x_\xi = M^{-1} \xi.
\]

One then defines \( \tilde{Q}(\xi) \) by the critical value of \( F_\xi \):

\[
\tilde{Q}(\xi) \overset{\text{def}}{=} F_\xi(x_\xi) = -\frac{1}{2} \langle \xi, M^{-1} \xi \rangle.
\]

One then obtains the representation \( W = \left\{ \left( x = -\nabla \tilde{Q}(\xi), \xi \right), \; \xi \in \mathbb{R}^d \right\} \).

More generally, we may want to “mix” the variables \( x \) and \( \xi \), namely choose a subset \( x' = (x_{i_1}, \ldots, x_{i_r}) \), the complementary subset \( x'' \) so that \( x = (x', x'') \), and similarly for \( \xi = (\xi', \xi'') \), and represent \( W \) as the graph of a function of \( 6 \quad (x', \xi'') \). How do things work? Inspired by the above case, we may perform a partial Legendre transform, and take, for a given \( x', \xi'' \), the function

\[
F_{x', \xi''}(x'') \overset{\text{def}}{=} Q((x', x'')) - \langle x'', \xi'' \rangle
\]

We may then look for the critical point \( x''_c \) of that function, and take \( \tilde{Q}(x', \xi'') = F_{x', \xi''}(x''_c) \).

**Exercise 2.10.** Compute \( x''_c \) and \( \tilde{Q}(x', \xi'') \), and check that the subspace \( W \) is then represented by

\[
(2.9) \quad W = \left\{ \left( x', x'' = -\partial_{x''} \tilde{Q}(x', \xi''); \xi' = \partial_{x'} \tilde{Q}(x', \xi''), \xi'' \right), \; (x', \xi'') \in \mathbb{R}^d \right\}.
\]

\( ^6 \)Notice that none of the components of \( x' \) and \( \xi'' \) are conjugate to each other: we want the “plane” \( (x', \xi'') \) itself to be Lagrangian.
Claim 2.11. Basic linear algebra shows that it is always possible to select an index $0 \leq r \leq d$ and a set of $r$ coordinates $x' = (x_1, \ldots, x_r)$, such that the subspace $W$ projects well on the coordinates $(x', \xi'')$: $W = \{(x, \xi), (\xi', x) = M(x', \xi'')\}$. Exercise: Find the conditions on $M$ for $W$ to be Lagrangian.

2.2.3. Generating functions of Lagrangian submanifolds. Quite surprisingly, any nonlinear Lagrangian submanifold can also be represented by a generating function. Take $\Lambda \subset \mathbb{R}^{2d}$ a $d$-dimensional submanifold (possibly with boundaries), and assuming $\Lambda$ can be written as

$$\Lambda = \left\{(x, \xi = F(x)), x \in U \subset \mathbb{R}^d\right\},$$

with $U$ some open set of $\mathbb{R}_x^d$ and $F : U \to \mathbb{R}_x^d$ smooth, then one can show TD 2 that $\Lambda$ is Lagrangian if and only if the function $F$ is a perfect gradient, that is, there exists a smooth function $S = S_\Lambda : U \to \mathbb{R}$ such that $F = \nabla S$, and hence

$$(2.10) \quad \Lambda = \left\{(x, \xi = \nabla S(x)), x \in U \subset \mathbb{R}^d\right\}.$$  

Such a function $S_\Lambda$ (unique up to an additive constant) is called a generating function for $\Lambda$. This representation allows to absorb all the constraints on the function $F$ due to the Lagrangian property. It is quite a dramatic simplification: all the information on $\Lambda$, originally encoded in a vector valued function $F : \mathbb{R}^d \to \mathbb{R}^d$, is now condensed in a scalar function $S : \mathbb{R}^d \to \mathbb{R}$.

Like in the linear case, it is generically possible to represent $\Lambda$ (at least locally) in various coordinate planes. The transformation from a generating function $S(x)$ to a generating function $\tilde{S}(\xi)$ can also be performed through a Legendre transform similar with the linear case. Namely, if $\Lambda$ can be represented as $(2.10)$ but does also project well on the $\xi$ variables (more precisely on the range $V = \nabla S(U)$), we may define $F_\xi = S(x) - \langle \xi, x \rangle$ and look for its critical points $x_\xi \in U$. These points solve $\partial_\xi S(x_\xi) = \xi$, so by our assumption on $W$ there exists a unique solution $x_\xi \in U$, corresponding to the unique point $(x_\xi, \xi) \in \Lambda$ with momentum coordinate $\xi$. If we now take $\tilde{S}(\xi) = S(x_\xi) - \langle x_\xi, \xi \rangle$, we find that it indeed generates $\Lambda$:

$$\partial_\xi \tilde{S}(\xi) = \frac{\partial x_\xi}{\partial \xi} \left( \frac{\partial S}{\partial x}(x_\xi) - \xi \right) - x_\xi = -x_\xi,$$

where we used the equation defining $x_\xi$.

The difference compared with the linear case, is that a given Lagrangian manifold may not be representable by a single generating function (and by a single set of coordinates). Indeed, the projection of $\Lambda$ to the (say) $x$-plane may become singular at a caustic (e.g. a fold). Near such a caustic, one has to use a different set of coordinates to represent $\Lambda$. But the projection of $\Lambda$ on this second set of coordinates may become singular on another part of $\Lambda$, etc. So, generally one needs several generating functions to represent a given Lagrangian manifold. The various representations may be connected to each other through Legendre transforms along the “generic” parts of $\Lambda$ which admit all of them.

Remark 2.12. Lagrangian submanifolds are crucial objects in semiclassical analysis. In particular, we will soon associate to such a manifold $\Lambda$ families $(u(h))$ of $h$-dependent states, called Lagrangian states (or also WKB-states, for Wentzell-Kramers-Brillouin, who first introduced such states) TD 3 .

Another reason to focus on Lagrangian submanifolds is through their connection with symplectic transformations.
2.3. Symplectic transformations.

**Definition 2.13.** A diffeomorphism \( \kappa : \mathbb{R}^{2d} \to \mathbb{R}^{2d} \) which preserves the symplectic form is called a symplectomorphism, or a canonical transformation, or a symplectic transformation.

**Example 2.14.** We have proved in Prop. 2.7 that for each \( t \), the Hamiltonian flow \( \Phi^t_H \) is a symplectic transformation (assuming it is defined in some neighbourhood).

A sloppy way to write this property is to write in coordinates\(^7\) \( \kappa(y, \eta) = (x(y, \eta), \xi(y, \eta)) \). Then, invariance means that, when expanding the form \( \sum_k d\xi_k \wedge dx_k \) in terms of \( d\eta_j \) and \( dy_j \), we find exactly

\[
\sum_j d\xi_j \wedge dx_j = \sum_k d\eta_k \wedge dy_k.
\]

A Darboux coordinate frame \((y_k, \eta_k)\) is a coordinate frame in which the symplectic form takes the form (2.11). Equivalently, this frame is obtained from the standard frame \((x, \xi)\) through a symplectomorphism.

**Remark 2.15.** From the symplectic form \( \omega \) one can cook up a natural volume form

\[
d\text{vol} = \frac{\omega^n}{n!}
\]
on the phase space. This volume form is obviously preserved by any symplectomorphism, showing that symplectomorphisms are conservative (or volume preserving).

2.3.1. Linear symplectic transformation. If we assume that the transformation \( \kappa \) is linear, we may write it in a matrix form

\[
\kappa \begin{pmatrix} y \\ \eta \end{pmatrix} = \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y \\ \eta \end{pmatrix}.
\]

Let us call \( M_\kappa \) the matrix.

**Proposition 2.16.** The matrix \( M = M_\kappa \) is symplectic iff the following conditions are satisfied:

\[
tAC, tBD \text{ are symmetric, and } tAD - tCD = I_d.
\]

The ensemble of (real) symplectic matrices on \( \mathbb{R}^{2d} \) form the symplectic group \( \text{Sp}(d, \mathbb{R}) \).

**Proof.** One can do explicit computations in coordinates. They are pretty cumbersome. A more elegant and simple proof is as follows.

Recall that \( \omega(V, W) = \langle JV, W \rangle \). Then, we write

\[
M^*\omega(V, W) \overset{\text{def}}{=} \omega(MV, MW) = \langle MV, MW \rangle = \langle tMJMV, W \rangle.
\]

We want the RHS to be equal to \( \omega(V, W) = \langle JV, W \rangle \). Since this should be true for any \( V, W \), it means (from the nondegeneracy of the scalar product) that the matrix \( M \) must satisfy the following equation:

\[
J = tMJM
\]

Expanding this property yields the proposition. \( \square \)

\(^7\)We will try to keep the convention that \((y, \eta)\) will denote the initial point, and \((x, \xi)\) the final (or image) point.
A nonlinear symplectomorphism $\kappa$ is symplectic iff, for each $\rho$, the matrix $D\kappa(\rho)$, representing the tangent map at $\rho$ (written in the standard coordinates), is symplectic.

2.3.2. A few examples of symplectic transformations. Let us give some instructive examples of symplectic transformations.

**Example 2.17.** Consider a “point transformation” on $\mathbb{R}^d$, namely a diffeomorphism $y \mapsto x = f(y)$. This transformation can be lifted onto a symplectomorphism on $\mathbb{R}^{2d}$:

$$
\kappa(y, \eta) = \left( x = f(y), \xi = (^tDf(y))^{-1}\eta \right).
$$

The linearization of such a symplectomorphism corresponds to block diagonal matrices $M = \begin{pmatrix} A & 0 \\ 0 & ^tA^{-1} \end{pmatrix}$, $A = Df(y)$ invertible.

**Exercise 2.18.** Prove that this lifted point transformation $\kappa$ is symplectic. A direct calculation of $D\kappa(\rho)$ is not so easy. One trick consists in looking for $\eta$ as a function of $y$ and $\xi$ (which will appear clear when we deal with generating functions).

Let us now consider more general symplectic matrices.

**Example 2.19.** The “phase space rotation by $\pi/2$”, $M = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, is symplectic. It exchanges (up to a sign) position and momentum variables.

A diffeomorphism $\kappa$ can be characterized by its graph

$$
\Gamma_{\kappa} = \{ (x, y; \xi, \eta) : (x, \xi) = \kappa(y, \eta), (y, \eta) \in \mathbb{R}^{2d} \},
$$

which is a $2d$-submanifold of $\mathbb{R}^{4d}$. We can see $\mathbb{R}^{4d}$ as a symplectic space, with symplectic structure

$$
\tilde{\omega} = \sum_j d\xi_j \wedge dx_j + d\eta_j \wedge dy_j.
$$

(equivalently, $\mathbb{R}^{4d}$ is the product of the two symplectic spaces $\mathbb{R}^{2d}_{x,\xi} \times \mathbb{R}^{2d}_{y,\eta}$). From the graph $\Gamma$ we construct the twisted graph $\Gamma'_{\kappa}$ by inverting the value of the “initial momentum”:

$$
\Gamma'_{\kappa} = \{ (x, y; \xi, -\eta) : (x, \xi) = \kappa(y, \eta), (y, \eta) \in \mathbb{R}^{2d} \}.
$$

**Proposition 2.20.** The diffeomorphism $\kappa$ is a symplectomorphism if and only if $\Gamma'_{\kappa}$ is a Lagrangian submanifold of $(\mathbb{R}^{4d}, \tilde{\omega})$. TD 2

2.3.3. Generating functions for symplectic transformations. We can now put together the characterization of a symplectic transformation through its twisted graph $\Gamma'_{\kappa}$, with the fact that any Lagrangian in the double space $\mathbb{R}^{4d}$ admits a generating function. As before, one first needs to select an appropriate set of $2d$ variables where $\Gamma'_{\kappa}$ projects well. This set is not obvious to choose. We know that $\Gamma'_{\kappa}$ projects well on the variables $(x, \xi)$, but this set does not form a Lagrangian subspace, so this projection is inappropriate. Let us look at the few examples of symplectomorphisms we’ve seen above.

---

Footnote: Below we order the coordinates by first giving the positions (final and initial), then the momenta (final and initial).
Example 2.21. The lifted point transformation (2.14) can obviously not be represented as a graph over the variables \((x, y)\). On the other hand, due to the invertibility of the tangent map \((Df(y))^{-1}\), it is possible to extract the initial momentum \(\eta\) can be extracted from the knowledge of \(y\) and \(\xi\) through
\[
\eta = (Df(y))^{-1}(\xi)\cdot (y, \xi) \in \mathbb{R}^{2d}.
\]
What could be a generating function in this representation? We want a function \(S(\xi, y)\) such that \(x = -\partial_\xi S\) and \(-\eta = \partial_y S\). The obvious choice is
\[
S(\xi, y) = -\langle \xi, f(y) \rangle.
\]

Exercise 2.22. By exchanging \(f\) with \(f^{-1}\), we inverse the time, and thus exchange final and initial coordinates. Hence we can as well represent \(\Gamma'_\kappa\) in the variables \(\langle x, \eta \rangle\). Write down the formula for a generating function \(S(x, \eta)\) in this case.

The representation in terms of variables \((\xi, y)\) is also convenient when one considers a Hamiltonian flow \(\Phi_t^\kappa\) for short times, that is when \(\Phi_t^\kappa\) is close from the identity map. Indeed, for the identity \(\kappa = Id\) neither \((x, y)\) nor \((\xi, \eta)\) work, but the coordinates \((\xi, y)\) are perfectly fine; the generating function for the identity (which is a particular case of point transformation) reads \(S_I(\xi, y) = -\langle \xi, y \rangle\). By continuity, the representation persists for a small deformation \(\Phi_t\) of the identity, so that one is able to define a generating function \(S_t(\xi, y)\) associated with the map \((x, \xi) = \Phi_t(y, \eta)\), such that \(S_{t=0}(\xi, y) = -\langle \xi, y \rangle\). It turns out that a solution for this problem is given by the action functional of the classical orbit \(\{ (x(s), \xi(s)) \}_{s \in [0, t]}\) such that \(x(0) = y\) and \(\xi(t) = \xi\) (for short times such an orbit exists, and is unique):
\[
S_t(\xi, y) = \int_0^t \langle \xi(s), \dot{x}(s) \rangle \, ds - \langle \xi, y \rangle.
\]

Example 2.23. On the opposite, the \(\pi/2\) rotation can be represented by a generating function of the type \(S(x, y)\). Indeed, the graph of the transformation \((x, \xi) = \kappa(y, \eta) = (-\eta, y)\) projects well on the coordinates \((x = -\eta, y)\), so we may look for a generating function \(S(x, y)\) such that \(\xi = \partial_x S(x, y)\), \(-\eta = \partial_y S(x, y)\). The obvious choice is \(S(x, y) = \langle x, y \rangle\).
3. (Semiclassical) quantizations on $\mathbb{R}^{2d}$

We now dwell a bit deeper into the quantization procedure mentioned in the introduction. We recall that this quantization allows to transform functions $a(x,\xi)$ (classical observables) into operators $\text{Op}_\hbar(a)$ acting on $L^2(\mathbb{R}^d)$ (or on smaller functional spaces dense in $L^2$, like the Schwartz space $S(\mathbb{R}^d)$).

**Definition 3.1.** The function $a(x,\xi)$ is often called the symbol of the operator $\text{Op}_\hbar(a)$. The symbol map is therefore the inverse of the quantization map$^{10}$. This notion depends of course on the type of quantization procedure. We will sometimes speak of right, left, Weyl symbol, in reference to these different quantizations.

We will later define the principal symbol of an operator, this notion being more robust to the choice of quantization.

**Remark 3.2.** We will see that one strength of the semiclassical formalism is to be able to describe the properties of operators $\text{Op}_\hbar(a)$ in terms of their symbols (functions on $\mathbb{R}^{2d}$), objects which are easier to handle than operators.

We know that a polynomial $p(x)$ is mapped to the multiplication operator by this polynomial, while a polynomial $p(\xi)$ is mapped into the differential operator $p(\hbar D_x)$, where from now on we use the notation

$$D = \frac{1}{i}\partial.$$

(notice that the operators $D$ are symmetric, while $\partial$ is skew-symmetric).

What is the use of this quantization? What form of operators are produced this way?

1. it allows to relate quantum observables $A : L^2 \to L^2$ to classical ones $a(x,\xi)$.
2. it allows obtain the quantum Hamiltonian (the generator of the Schrödinger flow) as a quantization of a classical Hamiltonian $P_\hbar = \text{Op}_\hbar(p)$, $p(x,\xi) = \frac{|\xi|^2}{2} + V(x)$.
3. it will create a class of operators which contain differential operators, but also a larger class of (semiclassical) pseudodifferential operators, which are useful in various matters. Even though the Hamiltonian is usually a differential operator, functions of it (e.g. its resolvent, or its noninteger powers when $P_\hbar$ is positive, functions $\theta(P_\hbar)$ useful when analyzing its spectrum) are pseudodifferential ops.
4. More generally, the class of operators produced by this quantization contains phase space cutoff operators, which are very useful to analyze the microlocalization properties of wavefunctions, that is, their localization properties both in position and momentum (Fourier) space. These cutoffs really make sense only in the semiclassical limit $\hbar \ll 1$.
5. Although the quantization can be defined for any value of $\hbar > 0$ (e.g. $\hbar = 1$), the theory becomes practical and quantitative essentially in the semiclassical limit $\hbar \to 0$, and this is the asymptotic regime we’ll be considering. The reason is, the objects (wavefunctions / Schwartz kernels of operators) develop fast oscillatory phases in this limit, which allows to use nonstationary phases or stationary phase approximations of relevant integrals, leading to compactly expressed asymptotics.

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$^{10}$We will see that the quantization is indeed an invertible procedure, at least formally.
3.1. Quantizations of symbols in $S(\mathbb{R}^{2d})$. In subsection 1.2.2 we have explained the required properties of a quantization. In order to give a unifying framework for both operators $\text{Op}_\hbar(f = f(x))$ and $\text{Op}_\hbar(g = g(\xi))$, we will write these operators in a similar form (for functions $\psi \in S(\mathbb{R}^d)$):

$$\text{Op}_\hbar(f) \psi(x) = f(x) \psi(x)$$

$$= \int e^{i\xi x/H} f(x) (\mathcal{F}_\hbar \psi) (\xi) \frac{d\xi}{(2\pi\hbar)^{d/2}}$$

$$= \iint e^{i\xi \cdot (y-x)/\hbar} f(x) \psi(y) \frac{d\xi \, dy}{(2\pi\hbar)^d}. \quad (3.1)$$

In the last line the integral is not absolutely convergent, but uses the representation of the delta distribution as an oscillatory integral:

$$\delta(x) = \int_{\mathbb{R}^d} e^{i\xi x} \frac{d\xi}{(2\pi\hbar)^d}.$$ 

**Exercise 3.3.** Prove this integral representation, by adding the integrable factor $e^{-\epsilon \xi^2}$ in the integrand, and letting $\epsilon \to 0$.

The quantization of the momentum function $g = g(\xi)$ can be represented similarly:

$$\text{Op}_\hbar(g) \psi(x) = \mathcal{F}_h^{-1}(g(\xi) \mathcal{F}_h \psi(\xi))(x)$$

$$= \int e^{i\xi x/H} g(\xi) (\mathcal{F}_h \psi) (\xi) \frac{d\xi}{(2\pi\hbar)^{d/2}}$$

$$= \iint e^{i\xi \cdot (y-x)/\hbar} g(\xi) \psi(y) \frac{d\xi \, dy}{(2\pi\hbar)^d}.$$ 

Notice that the double integral is now absolutely convergent.

In these expressions, the classical observables $f(x)$, resp. $g(\xi)$, appear in the integrands. Since the factor $f(x)$ can be taken out of the integral, we notice that

$$\text{Op}_\hbar(f(x)) \text{Op}_\hbar(g(\xi)) \psi(x) = \iint e^{i\xi \cdot (y-x)/\hbar} f(x) g(\xi) \psi(y) \frac{d\xi \, dy}{(2\pi\hbar)^d}. \quad (3.2)$$

In view of this property, it seems natural to propose the following definition for the quantization of a general observable $a(x, \xi)$.

**Definition 3.4.** (standard or right quantization) Take a symbol $a(x, \xi) \in S(\mathbb{R}^{2d})$. Then its standard (semiclassical) quantization is defined by

$$\text{Op}_\hbar^R(a) \psi(x) \overset{\text{def}}{=} \int e^{i\xi x/H} a(x, \xi) (\mathcal{F}_h \psi)(\xi) \frac{d\xi}{(2\pi\hbar)^{d/2}} = \iint e^{i\xi \cdot (y-x)/\hbar} a(x, \xi) \psi(y) \frac{d\xi \, dy}{(2\pi\hbar)^d}.$$ 

Notice that the integral is now absolutely convergent. This operator $\text{Op}_\hbar^R(a)$ will also be denoted by $a(x, \hbar D)$.

In this definition we have implicitly chosen a certain ordering between multiplication and differentiation: comparing with (3.2) we see that for a product $a(x, \xi) = f(x) g(\xi)$, we have

$$\text{Op}_\hbar^R(f(x)g(\xi)) = \text{Op}_\hbar(f(x)) \text{Op}_\hbar(g(\xi)),$$
hence we put the differentiation operators on the right and the multiplication on the left. This is the reason why the standard quantization is called the right quantization.

We notice that the operator \( \text{Op}_\hbar(f) \) in (3.1) could have been written alternatively as follows:

\[
\text{Op}_\hbar(f) \psi(x) = f(x) \psi(x) = \int \int e^{i \frac{\xi(x-y)}{\hbar}} f(y) \psi(y) \frac{d\xi \, dy}{(2\pi \hbar)^d}.
\]

But now we would have, instead of (3.2), the identity

\[
\text{Op}_\hbar(g(\xi)) \text{Op}_\hbar(f(x)) \psi(x) = \int \int e^{i \frac{\xi(x-y)}{\hbar}} g(\xi) f(y) \psi(y) \frac{d\xi \, dy}{(2\pi \hbar)^d} \overset{\text{def}}{=} \text{Op}_\hbar^L(g(\xi) f(x)) \psi(x)
\]

This representation leads to an alternative quantization, namely the left quantization where multiplication precedes differentiation:

**Definition 3.5.** (Left quantization) Take a symbol \( a(x, \xi) \in \mathcal{S}(\mathbb{R}^{2d}) \). Then its left (semiclassical) quantization is defined by

\[
\text{Op}_\hbar^L(a) \psi(x) \overset{\text{def}}{=} \int \int e^{i \frac{\xi(x-y)}{\hbar}} a(y, \xi) \psi(y) \frac{d\xi \, dy}{(2\pi \hbar)^d}.
\]

We notice that this cannot be simplified in terms of \( \mathcal{F}_\hbar \psi \), since we multiply \( \psi \) by \( f(y) \) before taking its Fourier transform.

The right (standard) and left quantizations are related to one another by duality.

**Lemma 3.6.** For any symbol \( a \in \mathcal{S}(\mathbb{R}^{2d}) \), we have the formal adjoint relation

\[
\text{Op}_\hbar^R(a)^* = \text{Op}_\hbar^L(\bar{a}).
\]

**Proof.** This formal duality means that we don’t pay attention to domains yet, but only consider test states \( \psi, \varphi \in \mathcal{S}(\mathbb{R}^{2d}) \). We then easily check that

\[
\langle \varphi, \text{Op}_\hbar^R(a) \psi \rangle = \iint dx \varphi(x) e^{i \frac{\xi(x-y)}{\hbar}} a(x, \xi) \psi(y) \frac{d\xi \, dy}{(2\pi \hbar)^d} = \iint dy \psi(y) \varphi(x) e^{i \frac{\xi(x-y)}{\hbar}} \bar{a}(x, \xi) \frac{d\xi \, dx}{(2\pi \hbar)^d} = \langle \text{Op}_\hbar^L(\bar{a}) \varphi, \psi \rangle.
\]

Here we applied the Fubini theorem, using the fact that the integral converges absolutely. \( \Box \)

**Claim 3.7.** In general the operators \( \text{Op}_\hbar^R(a) \) and \( \text{Op}_\hbar^L(a) \) are different from each other. Hence, for a real valued symbol \( a(x, \xi) \) the above Lemma shows that neither of them is formally selfadjoint.

This is the case in particular for symbols of the form \( a(x, \xi) = f(x) g(\xi) \), since we claim that the operators \( \text{Op}_\hbar(f(x)) \) and \( \text{Op}_\hbar(g(\xi)) \) generally do not commute with each other. This is easy to verify when we take for \( g(\xi) \) a polynomial (which is obviously not in the Schwartz space, but this is not very important).

**Example 3.8.** For instance, if \( g(\xi) = \xi_1 \), we get the easy commutator

\[
[hD_{x_1}, f(x)] = \frac{h}{i} \partial_{x_1} f(x), \quad \text{a multiplication operator}
\]
For \( g(\xi) = \xi_1 \xi_2 \), it gives
\[
hD_{x_1} hD_{x_2} f(x) = hD_{x_1} \left( f(x) hD_{x_2} + \frac{h}{i} \partial_{x_2} f(x) \right)
\]
\[
= [hD_{x_1}, f(x)] hD_{x_2} + f(x) hD_{x_1} hD_{x_2} + \frac{h}{i} [hD_{x_1}, \partial_{x_2} f(x)] + \frac{h}{i} \partial_{x_2} f(x) hD_{x_1}
\]
\[
\implies [hD_{x_1} hD_{x_2}, f(x)] = \frac{h}{i} \partial_{x_1} f(x) hD_{x_2} + \frac{h}{i} \partial_{x_2} f(x) hD_{x_1} - h^2 \partial_{x_1 x_2}^2 f(x),
\]
now a first order differential operator, which can be written as the right quantization of the symbol
\[
\frac{h}{i} \partial_{x_1} f(x) \xi_2 + \frac{h}{i} \partial_{x_2} f(x) \xi_1 - h^2 \partial_{x_1 x_2}^2 f(x).
\]
These examples are differential operators. If \( g(\xi) \) is a polynomial of degree \( n \), then \([g(hD_x), f(x)]\) will be a polynomial operator of degree \( n - 1 \), with the highest degree terms depending on the first derivatives of \( f \).

**Example 3.9.** For a Schwartz function \( g(\xi) \),
\[
g(hD_x)f(x)\psi(x) = \mathcal{F}_h^{-1} \left( g(\hat{f} \ast \hat{\psi})(\xi) \right) = \hat{g} \ast (f \psi)(x) = \int \hat{g}(x - y) f(y) \psi(y) dy,
\]
while
\[
f(x)g(hD_x)\psi(x) = \mathcal{F}_h^{-1} \left( (\hat{f} \ast \hat{\psi})(\xi) \right) = f \ast (\hat{g} \psi)(x) = \int f(x) \hat{g}(x - y) \psi(y) dy
\]
This lack of symmetry for real valued symbols is the main drawback of these two quantization, and the main reason why we will prefer to use a more symmetric quantization, namely the Weyl quantization defined as follows.

To introduce the Weyl quantization, we notice that the only difference between the left and right quantizations is that the symbol \( a \) in the Schwartz kernel is taken at the initial \((y)\), resp. final \((x)\) points. Why not take the symbol at some intermediate point between the two, namely at a point \(tx + (1-t)y\)? This convention leads to a continuous family of quantizations, defined as
\[
(3.3) \quad \text{Op}_t(a)\psi(x) \overset{\text{def}}{=} \iint e^{\frac{\xi(x-y)}{\hbar}} a(tx + (1-t)y, \xi) \psi(y) \frac{d\xi dy}{(2\pi \hbar)^d}.
\]
In particular we have the identifications \(\text{Op}_1^R(a) = \text{Op}_1(a)\) and \(\text{Op}_0^L(a) = \text{Op}_0(a)\). Besides, the above Lemma easily generalizes to the

**Lemma 3.10.** For any symbol \( a \in \mathcal{S}(\mathbb{R}^{2d}) \) and any \( t \in [0,1] \), we have the formal adjoint relation
\[
\text{Op}_t(a)^* = \text{Op}_{1-t}(\bar{a}).
\]
*Proof. Same as above.*

The Weyl quantization consists in taking the midpoint \( t = 1/2 \), that is \( \text{Op}_1^W(a) = \text{Op}_{1/2}(a) \).

**Definition 3.11.** *(Weyl quantization)* Take a symbol \( a(x, \xi) \in \mathcal{S}(\mathbb{R}^{2d}) \). Then its *(Wigner-)Weyl quantization* is defined by
\[
(3.4) \quad \text{Op}_h^W(a)\psi(x) \overset{\text{def}}{=} \iint e^{\frac{\xi(x-y)}{\hbar}} a \left( \frac{x + y}{2}, \xi \right) \psi(y) \frac{d\xi dy}{(2\pi \hbar)^d}.
\]
This expression does not seem very natural at first glance, but it leads to several nice properties specific to this quantization. From Lemma 3.10 we straightforwardly derive the following crucial property of the Weyl quantization.

**Lemma 3.12.** For any real-valued symbol \( a(x, \xi) \in \mathcal{S}(\mathbb{R}^d) \), the operator \( \mathcal{O}_h(W)(a) \) is formally selfadjoint.

We will see below that for such symbols the operator \( \mathcal{O}_h(W)(a) \) is bounded in \( L^2(\mathbb{R}^d) \), so that it is actually selfadjoint on \( L^2(\mathbb{R}^d) \).

### 3.2. Another route to the Weyl quantization.

We will recover the Weyl quantization from a different strategy, namely by using the phase space translation operators, also called Weyl-Heisenberg operators. These operators form a family of unitary operators on \( L^2 \), indexed by translation vectors \( (x_0, \xi_0) \in \mathbb{R}^2 \). They depend on Planck’s parameter \( h \), but this dependence will be omitted in the formulas. We will call \( T_{(x_0, \xi_0)} \), the operator performing the translation by the vector \( (x_0, \xi_0) \).

#### 3.2.1. The (Weyl-Heisenberg) phase space translation operators.

To define these operators, we start by purely spatial translations, namely the subclass of operators \( T_{(x_0,0)} \). Translating a state \( \psi \in L^2 \) by a space vector \( x_0 \) is an obvious operation:

\[
[T_{(x_0,0)}\psi](x) \overset{\text{def}}{=} \psi(x - x_0).
\]

Similarly, since momentum and position are exchanged by \( \mathcal{F}_h \), it looks obvious to define as follows the pure momentum translations:

\[
\mathcal{F}_h(T_{(0,\xi_0)}\psi)(\xi) = \mathcal{F}_h\psi(\xi - \xi_0)
\]

\[
\Rightarrow T_{(0,\xi_0)}\psi(x) = e^{i\frac{\xi_0 \cdot x}{h}} \psi(x).
\]

Hence, \( T_{(0,\xi_0)} \) is simply the multiplication operator by a linear phase function (corresponding to a plane wave of wavevector \( \xi_0/h \)).

Classically, a translation by the phase space vector \( V_0 = (x_0, \xi_0) \) is simply the combination of the translation by \( x_0 \) and by \( \xi_0 \), these translations forming the Galilean group. It thus sounds reasonable to take for \( T_{(x_0,\xi_0)} \) the product of the two preceding operators. However, the operators \( T_{(x_0,0)} \) and \( T_{(0,\xi_0)} \) do not commute, so one should (again) decide of a “best” ordering to define \( T_{(x_0,\xi_0)} \).

Let us look at the commutation properties:

\[
[T_{(x_0,0)}T_{(0,\xi_0)}\psi](x) = e^{i\frac{\xi_0(x-x_0)}{h}} \psi(x - x_0), \quad \text{while}
\]

\[
[T_{(0,\xi_0)}T_{(x_0,0)}\psi](x) = e^{i\frac{x_0 \cdot \xi}{h}} \psi(x - x_0),
\]

hence

\[
T_{(0,\xi_0)}T_{(x_0,0)} = e^{i\frac{\xi_0 \cdot x_0}{h}} T_{(x_0,0)}T_{(0,\xi_0)}
\]

Like above, it seems “natural” to define the joint translation \( T_{(x_0,\xi_0)} \) by selecting the “median point” between the two phases, and take:

\[
T_{(x_0,\xi_0)} \overset{\text{def}}{=} e^{i\frac{\xi_0 \cdot x_0}{2h}} T_{(x_0,0)}T_{(0,\xi_0)} = e^{-i\frac{\xi_0 \cdot x_0}{2h}} T_{(0,\xi_0)}T_{(x_0,0)}.
\]
This definition will be actually justified by the following remark. The multiplication operator \( T_{(0,\xi_0)} \) can obviously be defined by exponentiating the selfadjoint multiplication operator \( \text{Op}_\hbar(\xi_0 \cdot x) \):

\[
T_{(0,\xi_0)} = \exp \left( \frac{i}{\hbar} \text{Op}_\hbar(\xi_0 \cdot x) \right).
\]

Remark 3.13. This exponential operator can be obtained as the time-1 propagator \( e^{-iP/\hbar} \) of the Schrödinger equation, if we take for generator \( P = \text{Op}_\hbar(-\xi_0 \cdot x) \), which quantizes the linear Hamiltonian \( p(x, \xi) = -\xi_0 \cdot x \). Notice that the latter Hamiltonian generates the classical translation by \((0, \xi_0)\) at time 1.

Similarly, the space translation operator can be obtained by exponentiating the differential operator \( \text{Op}_\hbar(x_0 \cdot \xi) = \hbar x_0 \cdot \partial \):

\[
T_{(x_0,0)} = \exp \left( -\frac{i}{\hbar} \text{Op}_\hbar(x_0 \cdot \xi) \right) = \exp (-x_0 \cdot \partial_x).
\]

This fact can be proved by Fourier transform, or, as above, by solving the Schrödinger equation with Hamiltonian \( P_\hbar = \text{Op}_\hbar(x_0 \cdot \xi) \).

Now, because the classical translation \((x_0, \xi_0)\) is generated by the Hamiltonian \( p(x, \xi) = x_0 \cdot \xi - \xi_0 \cdot x \), it sounds natural to generate the corresponding quantum translation by the operator \( P_\hbar = \text{Op}_\hbar(x_0 \cdot \xi - \xi_0 \cdot x) \), and define the corresponding translation operator by the propagator:

\[
T_{(x_0,\xi_0)} \overset{\text{def}}{=} \exp \left( -\frac{i}{\hbar} \text{Op}_\hbar(x_0 \cdot \xi - \xi_0 \cdot x) \right) = \exp \left( -\frac{i}{\hbar} (x_0 \cdot hD_x - \xi_0 \cdot x) \right).
\]

Lemma 3.14. This definition of \( T_{(x_0,\xi_0)} \) exactly coincides with the “half phase” Ansatz.

Proof. Show that the Schrödinger equation

\[
i\hbar \partial_t \psi(t) = (x_0 \cdot hD_x - \xi_0 \cdot x) \psi(t)
\]

is solved by \( \psi(t, x) = e^{-it \frac{\xi_0 \cdot x}{2\hbar}} e^{it \frac{x_0 \cdot x}{\hbar}} \psi(x - tx_0) \).

Proposition 3.15. The family of operators \( T_{(x_0,\xi_0)} \) satisfy the following composition rules:

\[
T_{(x_0,\xi_0)}T_{(x_1,\xi_1)} = e^{i \frac{x_0 \cdot x_1 - x_0 \cdot \xi_1}{2\hbar}} T_{(x_0 + x_1, \xi_0 + \xi_1)}.
\]

Notice that the phase factor can be written in terms of the symplectic form:

\[
\xi_0 \cdot x_1 - x_0 \cdot \xi_1 = \omega(V_0, V_1), \quad \text{where} \ V_i = (x_i, \xi_i).
\]

Proof. Simple computation, using the formulas (3.7), which actually only depend on the commutation formula

\[
[\text{Op}(x_i), \text{Op}_\hbar(\xi_j)] = i\hbar \delta_{ij}.
\]

Remark 3.16. The commutation rules (3.9), as well as their “source” (3.10), are called the Heisenberg commutation relations. They show that the operators \( T_{(x_0,\xi_1)} \) form a unitary ray (or projective) representation of the Galilean group. These operators actually faithfully represent the Heisenberg group, a
noncommutative extension of the Galilean group, which includes an extra dimension to take into account the phase:

\[(x_0, \xi_0, s_0) \cdot (x_1, \xi_1, s_1) = (x_0 + x_1, \xi_0 + \xi_1, s_0 + s_1 + \frac{1}{2} (\xi_0 \cdot x_1 - x_0 \cdot \xi_1))\].

### 3.2.2. Microscopic translations = linear exponentials.

If we rescale the translation vectors by \( h \), we get

\[ T_{(hx_0, h\xi_0)} = \exp ( -i Op_h(x_0 \cdot \xi - \xi_0 \cdot x) ) = \exp ( -i(x_0 \cdot hD_x - \xi_0 \cdot x) ) \].

It seems natural to let the above operator define the quantization of the linear exponential functions

\[ e^{V_0}(x, \xi) \overset{\text{def}}{=} \exp (i(\xi_0 \cdot x - x_0 \cdot \xi)). \]

Notice that this choice corresponds to a choice of ordering between the operators \( hD_x \) and \( x \). The right (resp. left) quantization correspond instead to choosing

\[ Op^R_h(e^{V_0}) = T_{h(0,\xi_0)} T_{h(x_0,0)}, \quad Op^L_h(e^{V_0}) = T_{h(x_0,0)} T_{h(0,\xi_0)}. \]

**Lemma 3.17.** The quantization of \( e^{V_0} \) given by \( T_{(hx_0, h\xi_0)} \) exactly corresponds to the Weyl quantization (3.4). More generally, for any \( t \in [0,1] \) one has

\[ Op_t(e^{V_0}) = e^{-ih(1/2-t)\xi_0 \cdot x_0} T_{h(x_0,\xi_0)}. \]

**Proof.** We only need to compute the Weyl quantization of the exponential \( e^{V_0} \), using the formula (3.14):

\[ \text{(3.14)} \quad Op_t(e^{V_0}) \psi(x) = \int e^{i\frac{v(x-y)}{h}} \exp (i(\xi_0 \cdot (tx + (1-t)y) - x_0 \cdot \xi)) \psi(y) \frac{d\xi dy}{(2\pi h)^d} \]

\[ = e^{it\xi_0 \cdot x_0} \int e^{i\frac{v(x-y)}{h}} e^{-i\frac{v(\xi - h(1-t)\xi_0)}{h}} \psi(y) \frac{d\xi dy}{(2\pi h)^d} \]

\[ = e^{it\xi_0 \cdot x_0} \int \left[ \mathcal{F}_h \psi \right] (\xi - h(1-t)\xi_0) \frac{d\xi}{(2\pi h)^d/2} \]

\[ = e^{it\xi_0 \cdot x_0} \int \left[ \mathcal{F}_h \psi \right] (\xi - h(1-t)\xi_0) \frac{d\xi}{(2\pi h)^d/2} \]

\[ = e^{-ih(1-t)\xi_0 \cdot x_0} e^{i\xi_0 \cdot x} \psi(x - h\xi_0). \]

In the case \( t = 1/2 \) we exactly recover the expression (3.8) applied to the microscopic translation \( T_{h(x_0,\xi_0)} \).

By linearity, and using the **nonsemiclassical** Fourier transform we are able to define the Weyl quantization of any symbol \( a \in \mathcal{S}(\mathbb{R}^{2d}) \) (and, actually, any \( a \in \mathcal{S}'(\mathbb{R}^{2d}) \)) in terms of the translation operators.

Indeed, let us denote as follows the Fourier decomposition of \( a \in \mathcal{S}(\mathbb{R}^{2d}) \):

\[ a(x, \xi) = \int \exp (i(\xi_0 \cdot x - x_0 \cdot \xi)) \hat{a}(x_0, \xi_0) \frac{dx_0 d\xi_0}{(2\pi)^d} = \int \exp (i\omega(V_0, \rho)) \hat{a}(V_0) \frac{dV_0}{(2\pi)^d} \]

Compared with the standard definition of the Fourier transform \( \mathcal{F} a \), there is a sign change in the FT w.r.t. \( \xi \). Also, notice that \( x_0 \) is the Fourier parameter conjugate to \( \xi \), while \( \xi_0 \) is the Fourier parameter conjugate to \( x \). With this convention, we gather the following property

\[ a(x, \xi) = \int \exp (i(\xi_0 \cdot x - x_0 \cdot \xi)) \hat{a}(x_0, \xi_0) \frac{dx_0 d\xi_0}{(2\pi)^d} = \int \exp (i\omega(V_0, \rho)) \hat{a}(V_0) \frac{dV_0}{(2\pi)^d} \]
Proposition 3.18. The Weyl quantization of $a \in \mathcal{S}([\mathbb{R}^d])$ can be defined in terms of the translation operators as follows:

\begin{equation}
\text{Op}_h^{W}(a) = \int T_{h(x_0, \xi_0)} \hat{a}(x_0, \xi_0) \frac{dx_0 d\xi_0}{(2\pi)^d}.
\end{equation}

This formula now looks a bit less “arbitrary” than the original formula (3.4), since it directly originates from the group of phase space translation operators. This property provides this quantization very specific properties with respect to the action of quadratic operators and their exponentials.

3.3. Relationship between different quantizations. Using the expression $A \overset{\text{def}}{=} \text{Op}(a) = \int \text{Op}_h^{W}(e_{V_0}) \hat{a}(V_0) \frac{dV_0}{(2\pi)^d}$ and (3.13), we want to compute the symbol $a_t$ such that $A = \text{Op}_t(a_t)$, for $t \in [0, 1]$. Namely, we want the establish the connection between the $t$-symbol of an operator $A$ and its Weyl ($t = 1/2$)-symbol. Take $a \in \mathcal{S}([\mathbb{R}^d])$. Using the expression (3.13) we obtain

\begin{equation}
A = \text{Op}_t(a_t) = \int \text{Op}_h^{W}(e_{V_0}) e^{-i(t/2-\epsilon)\xi_0 \cdot x_0} \hat{a}_t(V_0) \frac{dV_0}{(2\pi)^d}.
\end{equation}

Since the Fourier decomposition is unique, this expression shows that the Fourier transforms of $a$ and $a_t$ are related as follows:

\[ \hat{a}(V_0) = e^{-i(t/2-\epsilon)\xi_0 \cdot x_0} \hat{a}_t(V_0). \]

More generally, the symbols $a_t$ and $a_s$ satisfy

\begin{equation}
\hat{a}_t(V_0) = e^{i\epsilon(s-t)\xi_0 \cdot x_0} \hat{a}_s(V_0).
\end{equation}

This expression shows that if $a_s \in \mathcal{S}$, then so does $a_t$. We also notice that, even if $a_{1/2}$ is defined independently of $\epsilon$, the symbols $a_s$ will explicitly depend on $\epsilon$. We now express this relation directly between $a_s$ and $a_t$.

Proposition 3.19. Assume $A = \text{Op}_t(a_t)$ for $t \in [0, 1]$, with $a_{1/2} \in \mathcal{S}([\mathbb{R}^d])$. We get the following (slightly formal) expression between the symbols $a_t$ and $a_s$:

\begin{equation}
a_t(x, \xi) = e^{i\epsilon(s-t)\partial_{x} \cdot \partial_{\xi}} a_s(x, \xi) = e^{i\epsilon(t-s)\partial_{x} \cdot \partial_{\xi}} a_s(x, \xi).
\end{equation}

Proof. In the integral equation (3.19) for $a_t(x, \xi)$, we express the extra factor $e^{i\epsilon(s-t)\xi_0 \cdot x_0}$ through a derivative of the integrand:

\begin{equation}
a_t(x, \xi) \overset{\text{def}}{=} \int e^{i(\xi_0 \cdot x_0 - \xi_0 \cdot x)} \hat{a}_t(x_0, \xi_0) \frac{dx_0 d\xi_0}{(2\pi)^d}
= \int e^{i\epsilon(s-t)\xi_0 \cdot x_0} e^{i(\xi_0 \cdot x_0 - \xi_0 \cdot x)} \hat{a}_s(x_0, \xi_0) \frac{dx_0 d\xi_0}{(2\pi)^d}
= \int e^{i\epsilon(s-t)\partial_{x} \cdot \partial_{\xi}} e^{i(\xi_0 \cdot x_0 - \xi_0 \cdot x)} \hat{a}_s(x_0, \xi_0) \frac{dx_0 d\xi_0}{(2\pi)^d}
= e^{i\epsilon(s-t)\partial_{x} \cdot \partial_{\xi}} a_s(x, \xi).
\end{equation}

These quadratic differentials will pop up regularly in the next sections. The computations below should appear as a preparation for the computations on the composition of \Psi DOs.
3.3.1. A first example of asymptotic expansion. What meaning should one give to an expression like (3.23), apart from its Fourier transform version? To avoid too cumbersome notations, we will take \( t = 1, s = 0 \). A naive expansion gives:

\[
a_1(x, \xi) = \sum_{j \geq 0} \frac{(i\hbar D_y \cdot D_\eta)^j}{j!} a_0(y, \eta) |_{y = x, \eta = \xi},
\]

which looks nice in the semiclassical regime, since terms are \( O(h^k) \). The only trouble is that this series is generally divergent for all values of \( \hbar \), since we have no a priori control on the growth when we increase the order of derivation, for instance the derivatives could grow much faster than \( j! \). Still, one can give a meaning to this series, as an asymptotic expansion.

**Definition 3.20.** Let \( (\psi(h))_{h \in [0,1]} \) be a family of elements in some Banach space \( B \), and let \( (\psi_j)_{j \in \mathbb{N}} \) be elements of the same space. We say that the family \( (\psi(h)) \) satisfies the asymptotic expansion

\[
(3.26) \quad \psi(h) \sim \sum_{j \geq 0} h^j \psi_j \quad \text{as } h \searrow 0,
\]

if for any \( N \geq 0 \), there exists \( C_N > 0 \) such that we have

\[
\left\| \psi(h) - \sum_{j=0}^{N-1} h^j \psi_j \right\|_B \leq C_N h^N, \quad \forall h \in (0,1].
\]

A similar definition holds for \( \psi(h), \psi_j \) elements of a Fréchet space \( F \) generated by a countable family of norms \( \| \cdot \|_\alpha \). Then for any \( N \geq 0 \) and for any \( \alpha \), there exists \( C_{\alpha,N} > 0 \) such that the corresponding inequality holds for the \( \alpha \)-norm. We will write \( \psi(h) = \sum_{j=0}^{N-1} h^j \psi_j + O(h^N)_B \), respectively \( \psi(h) = \sum_{j=0}^{N-1} h^j \psi_j + O(h^N)_F \).

**Proposition 3.21.** (Borel’s Theorem) Given any sequence \( (a_j \in F)_{j \geq 0} \), there exists a function \( a(h) : (0,1] \to F \) satisfying the asymptotic expansion (3.26). Two such functions \( a(h), \tilde{a}(h) \) satisfy \( a(h) = \tilde{a}(h) + O(h^\infty)_F \).

**Proof.** Let us first treat the case of a Banach space \( B \), with norm \( \| \cdot \| \). Choose a cutoff function \( \chi \in C_c^\infty[0, \infty) \) with \( \chi(t) = 1 \) on \([0,1]\) and \( \chi(t) = 0 \) for \( t \geq 2 \). We will select below a sequence \( \lambda_j \to \infty \), and consider the function

\[
a(h) \overset{\text{def}}{=} \sum_{j=0}^{\infty} h^j \chi(\lambda_j h) a_j.
\]

Since \( \lambda_j \to \infty \), for any \( h \in (0,1] \) the above series contains finitely many nonzero terms, so that \( a(h) \) is well-defined. We want some control on the decay of the terms. The idea is to let \( \lambda_j \) grow sufficiently fast, such that the terms \( h^j \chi(\lambda_j h) \|a_j\| \) decay uniformly when \( h \to 0 \). We just notice that

\[
h^j \chi(\lambda_j h) \|a_j\| = h^j \chi(\lambda_j h) \frac{\lambda_j h}{\lambda_j} \|a_j\| \leq h^{j-1} \frac{2}{\lambda_j} \|a_j\|.
\]

\[^{11}\text{To get such a control on high derivatives one needs some analyticity condition on } a, \text{ or at least Gevrey type regularity.}\]
If we assume iteratively
\[ \lambda_j > \max \left( 2^{j+1} \| a_j \|, \lambda_{j-1} + 1 \right), \]
we obtain the uniform bound
\[ h^j \chi(\lambda_j h) \| a_j \| \leq h^{j-1}/2^j, \quad \forall j \geq 0. \]
For each given \( n > 0 \) we want to control
\[ (3.27) \quad \left\| a(h) - \sum_{j=0}^n h^j a_j \right\|_s \leq \sum_{j=0}^\infty h^j (\chi(\lambda_j h) - 1_{j \leq n}) \| a_j \|_s, \]
On the one hand, for \( h > \lambda_n^{-1} \), by continuity of \( a(h) \) we may always find a constant \( C_n \) such that
\[ \left\| a(h) - \sum_{j=0}^n h^j a_j \right\| \leq C_n^h h^{n+1} \quad h \in [\lambda_n^{-1}, 1]. \]
On the other hand, for \( h < \lambda_n^{-1} \) the sequence in the RHS of (3.28) will start at the order \( j = n + 1 \), and is equal to
\[ \sum_{j=n+1}^\infty h^j \chi(\lambda_j h) \| a_j \|_s \leq h^{n+1} \| a_{n+1} \| + \sum_{j=n+2}^\infty h^{j-1}/2^j \leq h^{n+1} (\| a_{n+1} \|_s + 1). \]
Let us now treat the case of a Fréchet space \( \mathcal{F} \), the topology of which is defined by seminorms \( (\| \cdot \|_s)_{s \in \mathbb{N}} \).
To select the \( \lambda_j \) we proceed by a diagonal argument. Namely, we choose \( \lambda_j \) such as to ensure that the property
\[ h^j \chi(\lambda_j h) \| a_j \|_s \leq h^{j-1}/2^j \]
holds for all seminorms with \( s \leq j \).
This can be achieved by taking
\[ \lambda_j > \max \left( 2^{j+1} \max_{s \leq j} \| a_j \|_s, \lambda_{j-1} + 1 \right). \]
Now, given a seminorm \( s \) and an index \( n \), let us first assume that \( n \geq s \). Then, in the case \( h < \lambda_n^{-1} \) we have
\[ (3.28) \quad \left\| a(h) - \sum_{j=0}^n h^j a_j \right\|_s \leq \sum_{j=0}^\infty h^j (\chi(\lambda_j h) - 1_{j \leq n}) \| a_j \|_s \]
\[ (3.29) \quad \leq \sum_{j=n+1}^\infty h^j \chi(\lambda_j h) \| a_j \|_s \]
\[ (3.30) \quad \leq h^{n+1} (\| a_{n+1} \|_s + 1). \]
As before, the case of \( h > \lambda_n^{-1} \) is easy, one just needs to taking a large enough constant \( C_{s,n} \).

We now treat the case \( n < s \): we simply decompose
\[ a(h) - \sum_{j=0}^n h^j a_j = a(h) - \sum_{j=0}^s h^j a_j + \sum_{j=n+1}^s h^j a_j, \]
and the above result for the order $s$ estimate gives
\[ \| a(h) - \sum_{j=0}^{n} h^j a_j \| \leq C_{s,s} h^{s+1} + \sum_{j=n+1}^{s} h^j \| a_j \| \leq \max \left( C_{s,s}, \max_{j \leq s} \| a_j \| \right) h^{n+1}. \]

Let us now go back to the formal expansion (3.25). To show that it makes sense as an asymptotic expansion, we use a **Taylor expansion with integral remainder** of the exponential function$^{12}$:
\[ e^{ihD_y \cdot D_\eta} a_0 = \sum_{j=0}^{N-1} \frac{(ihD_y \cdot D_\eta)^j}{j!} a_0 + \frac{(ih)^N}{(N-1)!} \int du (1-u)^{N-1} (D_y \cdot D_\eta)^N e^{ihD_y \cdot D_\eta} a_0. \]

We need to prove that the last integral is $O(1)_S$ uniformly in $h$. In Fourier transform, this integrand corresponds to the multiplication of $\hat{a}_0(x_0, \xi_0)$ by a factor $(-\xi_0 \cdot x_0)^N (1-u)^{N-1} e^{-ih\xi_0 \cdot x_0}$. Since $\hat{a}_0 \in S$, multiplication by this factor keeps the symbol in a bounded set of $S$, uniformly in $h$ and $u \in [0,1]$ (this means that any seminorm is bounded uniformly in $h,u$). These uniform bounds obviously hold for the integrated function. This shows that the last term above, when applied to $a_s$, gives a term $O(h^N)_S$, and proves the asymptotic expansion
\[ a_1 = \sum_{j=0}^{N-1} h^j \frac{(iD_y \cdot D_\eta)^j}{j!} a_0 + O(h^N)_S. \]

It is easy to check that for general $(t,s)$ we get
\[ a_t = \sum_{j=0}^{N-1} h^j \frac{(i(t-s)D_y \cdot D_\eta)^j}{j!} a_s + O(h^N)_S. \]

The above proof of the asymptotic expansion combines several elements: an exponentiated differential, Fourier transform, and the **Taylor expansion with integral remainder**.

3.3.2. **Playing again with quadratic exponentials: an example of (quadratic) stationary phase expansion.**

We now give a third expression for $a_t$ as function of $a_s$. Starting from (3.24), if we expand $\tilde{a}_s(V_0)$ in function of $a_s(y,\eta)$, we get
\[ a_t(x,\xi) = \iint e^{i(h(t-s)\xi_0 \cdot x_0 + i(\xi_0 \cdot (x-y) - \xi_0 \cdot (\xi_0 \cdot \eta) - \eta))} a_s(y,\eta) \frac{dV_0 d\rho}{(2\pi)^{2d}}. \]

Now, we are able to integrate the imaginary Gaussian expression over $V_0$, using the following

**Lemma 3.22.** Let $Q$ be a nonsingular symmetric $n \times n$ real valued matrix. Then, the function $e^{\frac{i}{2}(x,Qx)} S'(\mathbb{R}^n)$ admits the following Fourier transform:
\[ \mathcal{F}_1 \left( e^{\frac{i}{2}(x,Qx)} \right)(\xi) = \frac{e^{i\pi \text{sgn}(Q)/4}}{|\text{det} Q|^{1/2}} e^{-\frac{1}{2} \langle \xi, Q^{-1} \xi \rangle}, \]

where $\text{sgn}(Q)$ denotes the signature of $Q$, that is the difference between the numbers of positive and negative eigenvalues.

\[ ^{12} f(h) = \sum_{j=0}^{N-1} h^n \frac{f^{(j)}(0)}{j!} + \frac{h^N}{(N-1)!} \int_0^1 (1-s)^{N-1} f^{(N)}(sh) ds. \]
Proof. We first recall the case of the Fourier transform of a real Gaussian function: let $G$ be a definite positive $n \times n$ matrix:

$$\int e^{-\frac{1}{2} \langle x, G x \rangle} e^{-ix \cdot \xi} \frac{dx}{(2\pi)^n} = e^{-\frac{1}{2} \langle \xi, G^{-1} \xi \rangle} \frac{1}{(\det G)^{1/2}}.$$ 

If we deform $G$ so that it acquires an imaginary part, still keeping a positive definite real part, we get the same expression, where the square root of $\det G$ is obtained by analytic continuation. When $G = -iQ + \epsilon I$ and $\epsilon \searrow 0^+$, the expansion of this determinant over the eigenvalues of $Q$ gives $(\det(\epsilon - iQ))^{1/2} = \prod_j (\epsilon - i\lambda_j)^{1/2}$. If $\lambda_j > 0$ this converges to $e^{-\pi/4|\lambda_j|^{1/2}}$, while for $\lambda_j < 0$ this goes to $e^{+\pi/4|\lambda_j|^{1/2}}$. Putting back the phases in the numerator, we get (3.32). □

Let us now apply this Lemma to compute the integral in (3.31). The quadratic form in $(x_0, \xi_0)$ is given by the matrix $Q = h(s-t)\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, which has signature 0, determinant $|\det Q| = (h(s-t))^{2d}$, and inverse $Q^{-1} = (h(s-t))^{-1} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. The integral over $dV_0/(2\pi)^d$ thus produces

$$a_t(x, \xi) = \frac{1}{|s-t|^d} \int \frac{dyd\eta}{(2\pi)^d} e^{i\frac{(y-x)\cdot (y-\xi)}{2h(s-t)}} a_s(y, \eta) = \frac{1}{|s-t|^d} \int \frac{dyd\eta}{(2\pi)^d} e^{i\frac{\eta \cdot y}{2h(s-t)}} a_s(x+y, \xi+\eta),$$

where in the last equality we just shifted the integration variables.

The integral exhibits a fast oscillatory phase when $h \to 0$. In such a case, the usual thing to do to analyze this integral is to proceed to a **stationary phase expansion**, that is identify the stationary points of the phase, and expand the integral around these points. We will see that the expansion we obtain is exactly the asymptotic expansion shown in the previous subsection. Actually, the proof of the stationary phase expansion (when the phase is quadratic) exactly parallels the proof of last subsection.

3.4. (**Non**)-stationary phase expansions. Let us take this opportunity to introduce a crucial analytical tool of semiclassical analysis, namely nonstationary and stationary phase estimates. Generally speaking, the goal is to estimate precise integrals of the type

$$I(h) = \int a(x) e^{i\frac{\varphi(x)}{h}} dx,$$

where $x \in \mathbb{R}^n$, $a \in C_c^\infty(\Omega)$ for some bounded domain $\Omega \subset \mathbb{R}^n$, and $\varphi \in C^\infty(\mathbb{R}^n)$ (or $C^\infty(\Omega)$) is a phase function. This integral is strongly oscillatory when $h \to 0$, so we expect it to be small in this limit. The question is: “how small is it?”

The answer to this question will depend on the critical points of $\varphi$, that is the points $x_c \in \Omega$ such that $\nabla \varphi(x_c) = 0$.

3.4.1. **Nonstationary phase estimates.**

**Theorem 3.23.** Assume that $\varphi$ has no stationary point on $\Omega$. Then, for any $N \geq 0$, there exists $C_{N,\varphi,a} > 0$ such that

$$|I(h)| \leq C_{N,\varphi,a} \hbar^N, \quad \forall \hbar \in (0, 1].$$

This property is denoted shortly as

$$I(h) = O(\hbar^N).$$
Remark 3.24. A more precise estimate of the right hand side, with an explicit dependence on \( \varphi'(x) = \partial_x \varphi(x) \), is given in (3.35) below.

Proof. We use iteratively integration by parts. To do it cleverly, we use the differential operator\(^{13}\)

\[
L = \frac{\hbar}{i} \frac{\varphi'(x)}{|\varphi'(x)|^2} \cdot \partial_x, \quad \text{which satisfies } Le^{i\varphi(x)/\hbar} = e^{i\varphi(x)/\hbar}.
\]

We can then write

\[
I(h) = \int a(x) \left[ L^k e^{i\hbar x} \right] dx = \int \left[ tL^k a(x) \right] e^{i\hbar x} dx.
\]

where we used integration by parts \( k \) times. The transposed operator \( ^tL = -\frac{\hbar}{i} \partial_x \frac{\varphi'(x)}{|\varphi'(x)|^2} = -\frac{\hbar}{i} \left( \partial_x \frac{\varphi'(x)}{|\varphi'(x)|^2} \right) - \frac{\hbar}{i} \frac{\varphi''(x)}{|\varphi'(x)|^2} \cdot \partial_x \). Since \( \varphi' \) never vanishes, we get the estimate

\[
| ^tL^k a(x) | \leq C \hbar \left( \frac{|\varphi''(x)|}{|\varphi'(x)|^2} |a(x)| + \frac{1}{|\varphi'(x)|} |\partial a(x)| \right).
\]

Applying it again, we find

\[
| ^tL^2 a(x) | \leq C \hbar^2 \left( \left( \frac{|\varphi''(x)|^2}{|\varphi'(x)|^4} + \frac{|\varphi'''(x)|}{|\varphi'(x)|^3} \right) |a(x)| + \frac{|\varphi''(x)|}{|\varphi'(x)|^3} |\partial a(x)| + \frac{1}{|\varphi'(x)|^2} |\partial^2 a(x)| \right).
\]

Taking absorbing all the higher derivatives of \( \varphi(x) \) in the constant prefactor, but keeping the denominators, we get the pointwise estimate

\[
| ^tL^2 a(x) | \leq C_{n,\varphi'} \hbar^2 \sum_{j=0}^{2} \frac{|\partial^j a(x)|}{|\varphi'(x)|^{2-2j}}.
\]

An straightforward induction argument shows that for any \( k \geq 0 \),

\[
| ^tL^k a(x) | \leq C_{k,n,\varphi'} \hbar^k \sum_{j=0}^{k} \frac{|\partial^j a(x)|}{|\varphi'(x)|^{2k-2j}}.
\]

As a result, we get\(^{14}\) for any \( N \geq 0 \):

\[
|I(h)| \leq C_{N,n,\varphi'} \hbar^N \sum_{j=0}^{N} \left\| \frac{|\partial^j a|}{|\varphi'|^{2N-2j}} \right\|_{L^1}.
\]

This more precise estimate will be very helpful in the following. For instance, when deriving stationary phase estimates, it allows to take advantage of situations when \( \varphi'(x) \) vanishes at some critical point, but \( a(x) \) also vanishes up to some order at the same point. It will also allow to get fast decay for states of the form \( \text{Op}_b(a)\psi \), away from the support of \( \psi \) \( \square \)

3.4.2. Stationary phase estimates. We want to allow the phase \( \varphi(x) \) to admit isolated stationary points, and will assume that these stationary points are nondegenerate. This means that, at each stationary point \( x_c, \varphi'(x_c) = \partial_x \varphi(x_c) = 0 \), but the Hessian \( \varphi''(x_c) \) is a nonsingular matrix. Our aim is then to prove the following stationary phase theorem:

\(^{13}\)To alleviate notations we write \( \varphi' = \partial_x \varphi, \varphi'' = \partial_x^2 \varphi \) etc.

\(^{14}\)IMPORTANT: Here, and in many estimates below, the constants prefactors may change from line to line, although keeping the same notation.
admits the following asymptotic expansion. For any $a \in C^\infty_c (\Omega)$ for $\Omega$ a bounded domain in $\mathbb{R}^n$. Then the integral with oscillatory phase

$$I(h) = \int dx \ a(x) e^{-i(x,Qx)/h},$$

admits the following asymptotic expansion. For any $N \geq 0$, there exists $C_N$ (depending on $\Omega$, $Q$) such that

$$I(h) \sim \frac{(2\pi h)^{n/2} e^{i\pi \text{sgn}(Q)/4}}{|\text{det} \ Q|^{1/2}} \sum_{j=0}^N \frac{h^j}{j!} \left( \frac{\langle D, Q^{-1} D \rangle}{2i} \right)^{j} a|_{x=0} \leq C_N h^{N+n/2} \sum_{|\alpha| \leq 2N+n+1} ||\partial^\alpha a||_{L^1}.$$

Proof. Once again, we will make a little detour through the Fourier side. The integral $I(h)$ can be seen as the scalar product between the function $a(x)$ and the function $e^{-i(x,Qx)/h}$. Through Parseval’s formula, this is equal to the scalar products between their Fourier transforms. Using Lemma 3.22, this leads to

$$I(h) = \frac{h^{n/2} e^{i\pi \text{sgn}(Q)/4}}{|\text{det} \ Q|^{1/2}} \int e^{-\frac{i}{2} \langle \xi, Q^{-1} \xi \rangle} F_1 (a) (\xi) d\xi.$$

Now that $h$ is in the numerator of the exponential, it makes sense to expand the latter in powers of $h$. We can apply the following expansion of the exponential,$^{15}$

$$e^{it} - \sum_{j=0}^{N-1} \frac{(it)^j}{j!} \leq \frac{|t|^N}{N!},$$

so as to get

$$\left| e^{-i\pi \text{sgn}(Q)/4} \frac{|\text{det} \ Q|^{1/2}}{h^{n/2}} I(h) - \sum_{j=0}^{N-1} \frac{(h/2i)^j}{j!} \int (\langle \xi, Q^{-1} \xi \rangle)^j F_1 (a) (\xi) d\xi \right| \leq \frac{(h/2)^N}{N!} \int |\langle \xi, Q^{-1} \xi \rangle| N F_1 (a) (\xi) d\xi.$$

The right hand side can be estimated above by

$$C_N h^N \sum_{|\alpha| = 2N} |\xi^\alpha F_1 a(\xi)| d\xi \leq C_n' h^N \sum_{|\alpha| \leq 2N+n+1} ||\partial^\alpha a||_{L^1}.$$

Proof. To get a slightly more precise estimate we could apply instead the Taylor expansion with integral remainder
where we used the standard estimate
\[ \|F_1a\|_{L^1} \leq C_n \sum_{|\alpha| \leq n+1} \|\partial^\alpha a\|_{L^1}. \]

Now, the terms
\[
\int \left( \langle \xi, Q^{-1} \xi \rangle \right)^j F_1(a)(\xi) d\xi = \int F_1 \left( \left( \langle D, Q^{-1} D \rangle \right)^j a \right)(\xi) d\xi
= (2\pi)^{n/2} \left( \langle D, Q^{-1} D \rangle \right)^j a(0),
\]
which ends the proof of our stationary phase estimate. \(\square\)

Remark 3.27. We notice that the terms of the development can be computed in the following manner. Taylor expand \(a(x)\) around \(x = 0\), getting the formal sum
\[ a(x) = \sum_{k \geq 0} \frac{\langle a^{(k)}(0), x^k \rangle}{k!} \]
(here the bracket is between the \(k\)-tensor \(a^{(k)}\) and \(k\) copies of \(x\)). One can now compute each integral of the form \(\int dx_1 \cdots dx_k \ e^{i \langle \xi, Q x \rangle / \hbar} / \hbar^k\). Odd polynomials lead to zero (cf the parity of the quadratic exponential), while even polynomials lead to the appearance of the matrix \(Q^{-1}\). The explicit result for the derivative of order \(k = 2j\) is the \(j\)-term in (3.37).

To extend this stationary phase estimate to the case of a general Morse function \(\varphi\), we may proceed in two ways. The most straightforward way uses the Morse Lemma in order to transform the phase function to a quadratic phase.

Proposition 3.28. (Morse Lemma) Assume \(\varphi(x)\) has a nondegenerate critical point at \(x_0 \in \mathbb{R}^n\). Then there exists a change of coordinates \(\kappa : \text{neigh}(0) \to \text{neigh}(x_0)\), with \(\kappa(0) = x_0\), \(d\kappa(0) = Id\), such that
\[ \varphi(x) = \varphi_2 \circ \kappa^{-1}(x), \]
where \(\varphi_2(y) = \varphi(x_0) + \frac{1}{2} \langle y, \varphi''(x_0) y \rangle\) in the corresponding neighbourhood of \(y = 0\).

Proof. For simplicity of notations, let us translate \(x_0\) to the origin. The Taylor expansion of \(\varphi\) at \(x = 0\) can be written locally as
\[ \varphi(x) = \varphi(0) + \frac{1}{2} \langle x, \varphi''(0) x \rangle + \mathcal{O}(x^3). \]

Due to the nondegeneracy of \(\varphi''(0)\), we may write the RHS as
\[ \varphi(x) = \varphi(0) + \frac{1}{2} \langle x, Q(x)x \rangle, \]
where \(Q(x)\) is a symmetric nondegenerate matrix, smoothly dependent on \(x\), such that \(Q(0) = \varphi''(0)\). The trick now is to construct a diffeomorphism \(\kappa\) with the announced properties, such that \(\langle x, Q(x)x \rangle = \langle \kappa^{-1}(x), Q(0)\kappa^{-1}(x) \rangle\). We may try to do it with the Ansatz \(\kappa^{-1}(x) = A(x)x\), with \(A(x)\) an invertible matrix, smoothly dependent on \(x\), with \(A(0) = Id\). Hence, we need to solve (in \(A(x)\)) the problem
\[
\Gamma \dot{A}(x) Q(0) A(x) = Q(x).
\]
This problem is solved by inverting the function $F : A \mapsto iAQ(0)A$ defined on the space of $n \times n$ matrices, with images in the space of $n \times n$ symmetric matrices. To find a (right) inverse to this function near $A = Id$, we linearize the equation at $A = Id$. Namely, we notice that

$$F(I + \delta A) = Q(0) + \delta Q, \quad \delta Q = i\delta AQ(0) + Q(0)\delta A.$$ 

The differential map $\delta A \mapsto \delta Q$ is surjective, and admits as inverse $\delta A = \frac{1}{2}Q(0)^{-1}\delta Q$. The implicit function theorem shows that there exists a map $G : Q \mapsto A$ with $G(Q(0) + \delta Q) = I + \frac{1}{2}Q(0)^{-1}\delta Q$, such that $F \circ G = Id$. As a result, the problem (3.38) can be solved by a matrix $A(x) = G(Q(x))$ depending smoothly on $x$. □

This change of coordinates “straightens out” the coordinates, such as to absorb the nonquadratic part of $\varphi$ at $x = x_0$.

Equipped from this Morse Lemma and the quadratic stationary phase Theorem 3.26, we may write the integral

$$I(h) = \int a(x) e^{i\frac{\varphi(x)}{h}} \, dx = \int e^{i\frac{\varphi(x)}{h}} a \circ \kappa(y) \, |\det \kappa(y)| \, dy,$$

so we may apply the stationary phase estimate with the quadratic phase $\varphi_2$ to the amplitude $b(y) = a \circ \kappa(y) \, |\det \kappa(y)|$. This proves the expansion of Theorem 3.25.

**Remark 3.29.** This proof of the stationary phase estimate (3.36) is not very effective, it does not explicitly compute the coefficients $A_{2k}(x, D)$. One way to do so is to write the phase function as

$$\varphi(x) = \varphi_2(x) + g(x), \quad g(x) = \mathcal{O} \left( (x - x_0)^3 \right),$$

and then expand the exponential $e^{ig(x)/h} a(x) = \sum_{k \geq 0} \frac{(ig(x)/h)^k}{k!} a(x)$. Then, for each $k \geq 0$, the product $\frac{(ig(x)/h)^k}{k!} a(x)$ can be Taylor expanded at $x = x_0$, and each polynomial factor can be explicitly integrated over. A little power counting shows that the “dangerous” factor $h^{-k}$ does not ruin the asymptotic expansion. Indeed, this term is of order $h^{-k} \mathcal{O} \left( (x - x_0)^{3k} \right)$ when $x \to x_0$. If $k$ is even, integrating over the quadratic phase yields a result of order $h^{-k} h^{n/2} h^{3k/2} = h^{n/2 + k/2}$. If $k$ is odd, the lowest order term will come from integrating $h^{-k} \mathcal{O} \left( (x - x_0)^{3k+1} \right)$, and is therefore of order $h^{-k} h^{n/2} h^{(3k+1)/2} = h^{n/2 + (k+1)/2}$. The polynomials $A_{2j}(x, D)$ will then depend on the germs of the terms $\frac{(ig(x)/h)^k}{k!} a(x)$ at $x_0$, with the condition $k \leq j$ (k even) or $k + 1 \leq j$ (k odd).

### 3.5. Composition of pseudodifferential operators.

Once we have decided how to quantize classical observables, we want to know how the obtained operators are composed with each other. Namely, for two symbols $a, b$ in some symbol class, we want to understand the properties of the operator $\text{Op}_h(a) \circ \text{Op}_h(b)$.

If $a, b \in \mathcal{S}(\mathbb{R}^{2d})$, we know that their Schwartz kernels belong to $\mathcal{S}(\mathbb{R}^{d} \times \mathbb{R}^d)$, so that the kernel of $\text{Op}_h(a) \circ \text{Op}_h(b)$ does too. As such, through a Fourier transform the latter kernel can be associated with a symbol $c(x, \xi)$. The question is: can we directly compute the symbol $c$ from $a, b$?

We will first give an exact expression for $c$, using the expression in terms of translation operators.

In a second step, we will start doing some semiclassical analysis, to expand the expression for $c(x, \xi)$ in an asymptotic series in powers of $h$, using our main asymptotic tool, namely stationary phase estimates.
3.5.1. Exact expression of the composed symbols. Let us take $a, b \in \mathcal{S}(\mathbb{R}^d)$. For the moment we will treat an arbitrary $t$-quantization. Using the expression

$$A = \text{Op}_t(a) = \int \text{Op}_t(e_{V_0}) \frac{dV_0}{(2\pi)^d},$$

we get

$$A \circ B = \text{Op}_t(a) \text{Op}_t(b) = \int \text{Op}_t(e_{V_0}) \frac{dV_0}{(2\pi)^d} \int \text{Op}_t(e_{V_1}) \frac{dV_1}{(2\pi)^d}.$$ 

From there we get

$$A \circ B = \int \int \frac{dV_0 dV_1}{(2\pi)^{2d}} e^{i\hbar((1-t)\xi_0 \cdot x_1 - t\xi_1 \cdot x_0)} \hat{a}(V_0) \hat{b}(V_1) \text{Op}_t(e_{V_0 + V_1}),$$

where we used the change of variables

$$(V_0, V_1) \mapsto (V_+ = V_0 + V_1, V_- = \frac{1}{2}(V_0 - V_1)).$$

In these coordinates the phase reads

$$\varphi_t = (1-t)(\xi_+/2 + \xi_-) \cdot (x_+/2 - x_-) - t(\xi_+/2 - \xi_-) \cdot (x_+ / 2 + x_-)$$

$$= (1-t)(\xi_+ \cdot x_+/4 - \xi_- \cdot x_+/2 + \xi_- \cdot x_- / 2 - \xi_+ \cdot x_+/2 - \xi_- \cdot x_-) - t(\xi_+ \cdot x_+/4 + \xi_+ \cdot x_- / 2 - \xi_- \cdot x_+/2 - \xi_- \cdot x_-)$$

$$= (1-2t)(\xi_+ \cdot x_+/2 - 2\xi_- \cdot x_-) + (\xi_+ \cdot x_- + \xi_- \cdot x_+)/2.$$ 

We can identify the $t$-symbol of the operator $C = A \circ B$:

$$\hat{c}_t(V_+) = \int \frac{dV_-}{(2\pi)^d} e^{i\hbar((1-2t)(\xi_+ \cdot x_+/2 - 2\xi_- \cdot x_-) + \frac{1}{2}(-\xi_+ \cdot x_- + \xi_- \cdot x_+))} \hat{a}(V_+/2 + V_-) \hat{b}(V_+/2 - V_-).$$

Let us distinguish two cases:

1. In the case of the Weyl quantization ($t = 1/2$), the phase reads $\frac{1}{2}(\xi_- \cdot x_+ - \xi_+ \cdot x_-) = \frac{1}{2}\omega(V_-, V_+) = \frac{1}{2}\omega(V_0, V_1)$.

2. In the case of the standard (right) quantization ($t = 1$), the phase reads $-(\xi_+/2 - \xi_-) \cdot (x_+/2 + x_-) = -x_0 \cdot \xi_1$.

We will restrict ourselves to the Weyl quantization (and omit the subscripts $\bullet_{1/2}$). The above expression simplifies to

$$\hat{c}(V_+) = \int \frac{dV_-}{(2\pi)^d} e^{i\frac{\omega}{2}(V_-, V_+)} \hat{a}(V_+/2 + V_-) \hat{b}(V_+/2 - V_-).$$

Notice that $a, b \in \mathcal{S}$ imply that $c \in \mathcal{S}$ as well.
Can we get a decent expression of $c$ as a function of $a, b$? Expanding the (inverse) Fourier transforms, we get

$$c(\rho) = \int e^{i\omega(V_+ + \rho)} \hat{c}(V_+) \frac{dV_+}{(2\pi)^d}$$

$$= \int \frac{dV_- dV_+}{(2\pi)^{2d}} e^{i\omega(V_+ + \rho)} e^{i\frac{\hbar}{2} \omega(V_+ - V_+) \hat{a}_1/2} (V_+ / 2 + V_-) \hat{b}_{1/2} (V_+ / 2 - V_-)$$

$$= \int \int \frac{dV_- dV_+ d\rho_0 d\rho_1}{(2\pi)^{4d}} e^{i\omega(V_+ + \rho)} e^{i\frac{\hbar}{2} \omega(V_+ - V_+) e^{i\omega(\rho_0, V_+ / 2 + V_-) + i\omega(\rho_1, V_+ / 2 - V_-)} a(\rho_0) b(\rho_1)}$$

(3.40)

$$= \int \int \int \frac{dV_0 dV_1 d\rho_0 d\rho_1}{(2\pi)^{4d}} e^{i\omega(V_0 + V_1, \rho)} e^{i\omega(\rho_0, V_0) + i\omega(\rho_1, V_1)} a(\rho_0) b(\rho_1),$$

where we came back to the original variables $(V_0, V_1)$.

The phase in the exponential reads

$$h(JV_0, V_1)/2 + \langle J(V_0 + V_1), \rho \rangle - \langle JV_0, \rho_0 \rangle - \langle JV_1, \rho_1 \rangle \overset{\text{def}}{=} h(JV_0, V_1)/2 + \varphi(\rho_0, \rho_1)$$

We can apply the same Fourier-transmutation trick as in Proposition 3.19:

$$e^{-i\varphi} D_{\rho_i} e^{i\varphi} = -J V_i.$$

Hence we may replace the quadratic expression $h(JV_0, V_1)/2$ by derivatives in the $\rho_i$:

$$\frac{h}{2} \langle JV_0, V_1 \rangle = \frac{h}{2} \langle JJV_0, JV_1 \rangle = e^{-i\varphi} \frac{h}{2} \langle JD_{\rho_0}, D_{\rho_1} \rangle e^{i\varphi}$$

$$= e^{-i\varphi} \frac{h}{2} \omega(D_{\rho_0}, D_{\rho_1}) e^{i\varphi}.$$

The above integral then reads

$$c(\rho) = \int \int \int \frac{dV_0 dV_1 d\rho_0 d\rho_1}{(2\pi)^{4d}} e^{i\frac{\hbar}{2} \omega(D_{\rho_0}, D_{\rho_1})} \left[ e^{i\omega(V_0 + V_1, \rho)} e^{i\omega(\rho_0, V_0) + i\omega(\rho_1, V_1)} \right] a(\rho_0) b(\rho_1),$$

$$= \int \int \int \frac{dV_0 dV_1 d\rho_0 d\rho_1}{(2\pi)^{4d}} e^{i\omega(V_0 + V_1, \rho)} e^{i\omega(\rho_0, V_0) + i\omega(\rho_1, V_1)} e^{i\frac{\hbar}{2} \omega(D_{\rho_0}, D_{\rho_1})} [a(\rho_0) b(\rho_1)]$$

In the second line we performed an integration by parts on the variables $\rho_0, \rho_1$, and used the fact that the transpose of $\omega(D_{\rho_0}, D_{\rho_1})$ is equal to itself.

Now the integrals over $V_i$ are simple, they yield a product of $\delta$ functions: $(2\pi)^d \delta(\rho - \rho_0) \delta(\rho - \rho_1)$ In the end we obtain the simple formal equation

$$c(\rho) = e^{i\frac{\hbar}{2} \omega(D_{\rho_0}, D_{\rho_1}) a(\rho_0) b(\rho_1)} |_{\rho_0 = \rho_1 = \rho}.$$

**Theorem 3.30.** (composition of $\Psi$DOs). Assume $a, b \in \mathcal{S}(\mathbb{R}^{2d})$. Then the operator $\text{Op}_h^W(a) \circ \text{Op}_h^W(b) = \text{Op}_h^W(c)$, where the symbol $c \in \mathcal{S}(\mathbb{R}^{2d})$ is given by the formal expression

(3.41) 

$$c(\rho) = e^{i\frac{\hbar}{2} \omega(D_{\rho_0}, D_{\rho_1}) a(\rho_0) b(\rho_1)} |_{\rho_0 = \rho_1 = \rho} = e^{i\frac{\hbar}{2} (D_{\rho_0} - D_{\rho_1}) a(\rho_0) b(\rho_1)} |_{\rho_0 = \rho_1 = \rho}$$

We write $c = a \#_h b$, where the $\#_h$ is called the **Moyal product** of the symbols $a$ and $b$. This product can be rigorously defined on the Fourier side, namely by (3.39).
Exercise 3.31. Show that in the case of the right quantization, the composition formula reads, in terms of symbols,

\[ c_1(\rho) = e^{i\hbar D_{\delta_0}D_s}a_1(\rho_0)b_1(\rho_1)|_{\rho_0=\rho_1=\rho}. \]

Like in §3.3.2, the rather formal expression (3.41) can be given a meaning, either as an oscillatory convolution integral, or as an asymptotic expansion in powers of \( \hbar \).

Let us start again from the integral expression (3.40):

\[ c(\rho) = \iint_{\mathbb{R}^{2d}} \frac{dV_0dV_1d\rho_0d\rho_1}{(2\pi)^{2d}} e^{i\omega(V_0+V_1,\rho)} e^{\frac{i\hbar}{2}\omega(V_0,V_1)} e^{i\omega(\rho_0,V_0) + i\omega(\rho_1,V_1)} a(\rho_0)b(\rho_1). \]

Using Lemma 3.22, we easily integrate over the Fourier variables \( V = (V_0, V_1) \in \mathbb{R}^{2d} \):

\[
\iint_{\mathbb{R}^{2d}} \frac{dV_0dV_1}{(2\pi)^{2d}} e^{\frac{i\hbar}{2}(J\rho_0,J(\rho-\rho_0))} e^{-i\omega(V_1, J(\rho-\rho_0))} = \mathcal{F}_1 \left( e^{\frac{i\hbar}{2}V} \right) \left( \frac{\rho - \rho_0}{\rho - \rho_1} \right),
\]

\[
= \frac{1}{|\det Q|^{1/2}} \exp \left( -\frac{i}{2} \left( \frac{\rho - \rho_0}{\rho - \rho_1} \right) Q^{-1} \left( \frac{\rho - \rho_0}{\rho - \rho_1} \right) \right)
\]

with the symmetric matrix \( Q = \frac{1}{\hbar} \begin{pmatrix} 0 & -J \\ J & 0 \end{pmatrix} \). This matrix has signature \( 0 \), determinant \( |\det Q| = (\hbar/2)^d \) and inverse \( Q^{-1} = \frac{1}{\hbar} \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \). Hence we get

\[ c(\rho) = \iint_{\mathbb{R}^{2d}} \frac{d\rho_0d\rho_1}{(\pi\hbar)^{2d}} \exp \left( -\frac{2i\hbar}{\hbar} \left( J(\rho - \rho_0), (\rho - \rho_1) \right) \right) a(\rho_0)b(\rho_1) \]

\[ = \iint_{\mathbb{R}^{2d}} \frac{d\rho_0d\rho_1}{(\pi\hbar)^{2d}} \exp \left( -\frac{2i\hbar}{\hbar} \omega(\rho_0, \rho_1) \right) a(\rho + \rho_0)b(\rho + \rho_1) \]

after the change of variables \( \rho - \rho_1 = -\rho'_1 \).

Proposition 3.32. The Moyal product of two symbols \( a, b \in \mathcal{S}(\mathbb{R}^{2d}) \) can be expressed as the following oscillatory convolution integral:

\[ a\#_h b(\rho) = \iint_{\mathbb{R}^{2d}} \frac{d\rho_0d\rho_1}{(\pi\hbar)^{2d}} \exp \left( -\frac{2i\hbar}{\hbar} \omega(\rho_0, \rho_1) \right) a(\rho + \rho_0)b(\rho + \rho_1). \]

3.5.2. Asymptotic expansion of the composed symbol. Expanding the operator \( e^{\frac{i\hbar}{2}\omega(D_{\rho_0}, D_{\rho_1})} \) to finite order and using the Taylor estimate with integral remainder, we find\(^\text{16}\)

\[ a\#_h b(\rho) = a(\rho) e^{\frac{i\hbar}{2}\omega(\overline{D}, \overline{D})} b(\rho) \]

\[ = \sum_{j=0}^{N-1} \frac{(ih/2)^j}{j!} a(\rho) \left( \omega(\overline{D}, \overline{D}) \right)^j b(\rho) + \frac{(ih/2)^N}{(N-1)!} \int du (1-u)^{N-1} a(\rho) \left( \omega(\overline{D}, \overline{D}) \right)^N e^{iu\omega(\overline{D}, \overline{D})} b(\rho). \]

We just need to estimate the integral. Transforming to the Fourier side, we find that it is given by a linear combination of \( \mathcal{F}_1^{-1} \left( \omega(V_0, V_1) \right)^N e^{iu\omega(\overline{V}_0, \overline{V}_1)} a(V_0)b(\overline{V}_1) \), taken at the point \( \rho_0 = \rho_1 = \rho \). Since \( a, b \in \mathcal{S} \), then this Fourier transform is also in \( \mathcal{S}_{\rho_0, \rho_1} \), and therefore the full integral is in \( \mathcal{S}_\rho \). One can control the remainder in terms of the derivatives of \( a, b \):

\(^{16}\)We use here an abbreviated notation to denote derivatives acting on \( a \) or \( b \).
Lemma 3.33. For $a, b \in \mathcal{S}(\mathbb{R}^{2d})$, the seminorms of $a \#_{\hbar} b$ are controlled by

$$|\rho^\alpha \partial^\alpha (a \#_{\hbar} b) (\rho)| \leq \sum_{j=0}^{N-1} \frac{(\hbar/2)^j}{j!} \left| \rho^\alpha \partial^\alpha a \left( \omega(\vec{D}, \vec{D}) \right)^j b(\rho) \right| + C_{N, \gamma, \alpha} \hbar^N \left\| \langle \rho \rangle^{\gamma} | (D) |^{N+|\alpha|+2d+1} a \right\|_{L^1} \left\| (D) |^{N+|\alpha|+|\gamma|+2d+1} b \right\|_{L^1}.$$  

The norms in the last term could be symmetrized between $a$ and $b$. They can be replaced by norms of the type $\sum_{|\beta| \leq N+|\alpha|+2d+1} \| \langle \rho \rangle^{\gamma} | \partial^{\beta} a \|_{L^1}$.

For $N = 0$ this bound reads

$$\forall \alpha, \gamma \in \mathbb{N}^{2d}, \quad |\rho^\alpha \partial^\alpha (a \#_{\hbar} b) (\rho)| \leq C_{\gamma, \alpha} \left\| \langle \rho \rangle^{\gamma} | (D) |^{0+|\alpha|+2d+1} a \right\|_{L^1} \left\| (D) |^{0+|\alpha|+|\gamma|+2d+1} b \right\|_{L^1}.$$  

Proposition 3.34. The Moyal product satisfies the following asymptotic expansion:

$$a \#_{\hbar} b = \sum_{j=0}^{N-1} \frac{(i \hbar/2)^j}{j!} a(\rho) \left( \omega(\vec{D}, \vec{D}) \right)^j b(\rho) + O(\hbar^N).$$  

(3.43)

$$= a(\rho) b(\rho) + \frac{i \hbar}{2} (D_\xi a \cdot D_x b - D_x a \cdot D_\xi b) - \frac{\hbar^2}{8} a(\rho) \left( \vec{D}_\xi \cdot \vec{D}_x - \vec{D}_x \cdot \vec{D}_\xi \right)^2 b(\rho) + O(\hbar^3)$$

$$= a(\rho) b(\rho) - \frac{i \hbar}{2} \{a, b\} - \frac{\hbar^2}{8} a(\rho) \left( \left( \vec{D}_\xi \cdot \vec{D}_x \right)^2 + \left( \vec{D}_x \cdot \vec{D}_\xi \right)^2 - 2 \vec{D}_\xi \cdot \vec{D}_x \vec{D}_\xi \cdot \vec{D}_x \right) b(\rho) + O(\hbar^3).$$

The first term (0th order) represents the classical product of observables.

The second term ($\hbar^1$) is proportional to the Poisson bracket of the classical observables, which is obviously antisymmetric w.r.t. exchanging $a$ and $b$. This antisymmetry will be the case for all odd-order terms $\hbar^{2k+1}$, while the even-order terms (like the $\hbar^2$ term above) will be symmetric.

Corollary 3.35. (Commutator of $\Psi$DOs). For $a, b \in \mathcal{S}$, the commutator of the two quantizations can be written

$$[\text{Op}^W_h (a), \text{Op}^W_h (b)] = \text{Op}^W_h (a \#_{\hbar} b - b \#_{\hbar} a)$$

(3.44)

$$= \frac{\hbar}{i} \text{Op}^W_h (\{a, b\}) + O(\hbar^3).$$

The fact that the commutator of two operators is approximately represented by the quantization of the Poisson bracket is a very important property of quantization. It is actually satisfied by all the quantizations $\text{Op}_t$. The specificity of the Weyl quantization is the absence of a term $O(\hbar^2)$ in this expression.

Why is this connection so important?

Because the Poisson bracket, as we’ve seen, generates the classical evolution of observables. Indeed, we had found in (2.7) the expression for infinitesimal time evolution:

$$\{p, a\} = \frac{d}{dt} (a \circ \Phi^t_p)_{|t=0}.$$
On the other hand, the adjoint evolution of quantum observables, described by the Heisenberg evolution (5.1), gives infinitesimally
\[ \frac{i}{\hbar} [P, A] = \frac{d}{dt} A(t)|_{t=0}. \]
The semiclassical correspondence (3.44) then connects the quantum and classical evolutions of observables, up to a small semiclassical remainder:
\[ \frac{i}{\hbar} [P, A] = \frac{i}{\hbar} [\text{Op}_h^W(p), \text{Op}_h^W(a)] = \text{Op}_h^W \{\{a, b\}\} + \mathcal{O}(\hbar^2). \]

The Egorov Theorem we will show below, which exactly expresses the correspondence between the evolution of classical and quantum observables for finite (or large) times, is the “integrated form” of the above correspondence.

3.5.3. Operators with disjoint essential supports. One simple application of the asymptotic expansion (3.43) concerns the case of symbols a, b ∈ $S(\mathbb{R}^2d)$ with disjoint supports.

**Corollary 3.36.** Assume a, b ∈ $C_c^\infty(\mathbb{R}^2d)$. Then the operator $\text{Op}_h^W(a) \circ \text{Op}_h^W(b)$ is residual (meaning that its symbol is $\mathcal{O}(\hbar^\infty)_S$).

We can actually control the norm of estimate also in terms of the distance between $\text{supp } a$ and $\text{supp } b$, see Lemma ?? below.

The operators $\text{Op}_h^W(a)$, $\text{Op}_h^W(b)$ are “supported” on disjoint parts of phase space, so they are in some sense “orthogonal to each other”. The support of $a$ is here equivalent with the semiclassical wavefront set of the operator $\text{Op}_h(a)$. This property of pseudodifferential operators is sometimes called quasi-locality, by analogy with the locality of differential operators (if two differential operators $a(x, D)$ and $b(x, D)$ have disjoint supports, then $a(x, D) \circ b(x, D) = 0$).

3.6. Extending the quantization to nondecaying symbols. So far we have defined the quantization of symbols of the form $f(x), g(\xi)$, with $f, g \in S(\mathbb{R}^d)$, or $a \in S(\mathbb{R}^2d)$. Such fast decaying functions are useful in the course of analyzing phase space localization of quantum states (we will often use cutoff functions $\chi \in C_c^\infty(\mathbb{R}^2d)$). However, we also want to be able to quantize unbounded symbols, like the Hamiltonian $p(x, \xi) = |\xi|^2 + V(x)$, which is unbounded in $\xi$, but can also be unbounded in $\xi$.

We will see that the above quantizations $\text{Op}_h$ can be easily extended to certain classes of symbols with appropriate growth properties.

3.6.1. The class of uniformly bounded symbols $S(\mathbb{R}^2d)$, and dealing with oscillatory integrals. One useful class of symbols is the class $S(\mathbb{R}^2d)$ of smooth functions, with all derivatives uniformly bounded over $\mathbb{R}^2d$:

\[ S(\mathbb{R}^2d) \overset{\text{def}}{=} \left\{ a \in C^\infty(\mathbb{R}^2d), \ |\partial^\alpha_x \partial^\beta_\xi a(x, \xi)| \leq C_{\alpha, \beta}\right\}. \]

Below we will often not consider individual functions $a(x, \xi)$, but rather families of functions $a = (a_h)_{h\in[0,1]}$. By saying that $a \in (a_h) \in S(\mathbb{R}^2d)$, we mean that the bounds $C_{\alpha, \beta}$ are uniform over the full family.

For such symbols, even if we take $\psi \in S(\mathbb{R}^d)$, the integral
\[ I(\psi) = \iint e^{i(\xi(x-y)/\hbar)} a\,(tx + (1-t)y, \xi) \psi(y) \frac{d\xi dy}{(2\pi\hbar)^d}. \]
is absolutely convergent in the $y$ variable, but not in the $\xi$ variable. Still the presence of an oscillatory phase will help us to give a meaning to such a so-called oscillatory integral. Indeed, the strategy will be to apply sufficiently many “formal” integration by parts (without paying attention to the behaviour at the boundaries), in order to recover an absolutely convergent integral. To make this elegantly, for each $\xi$ we insert the differential operator $L_\xi \overset{\text{def}}{=} \frac{1+i\hbar\xi}{1+|\xi|^2}$, which satisfies the property

$$L_\xi e^\frac{i\xi (x-y)}{\hbar} = e^\frac{i\xi (x-y)}{\hbar}.$$  

Integration by parts means that we apply instead the transposed of this operator

$$I(\psi) = \iint \left( L_\xi e^\frac{i\xi (x-y)}{\hbar} \right) a(tx + (1-t)y, \xi) \psi(y) \frac{d\xi \, dy}{(2\pi\hbar)^d} \overset{\text{def}}{=} \iint \frac{e^{i\xi (x-y)} L_\xi (a(tx + (1-t) y, \xi) \psi(y))}{(2\pi\hbar)^d} dy \, d\xi.$$  

The action of $L_\xi$ on $a\psi$ differentiates the symbol $a$ and the state $\psi$, but the resulting product is still fast decaying in $y$. In the $\xi$ variable we have gained a factor $\frac{O(|\xi|)}{1+|\xi|^2} = O(|\xi|^{-1})$, where we have used the “Japanese brackets” notation

$$\langle \xi \rangle \overset{\text{def}}{=} (1 + |\xi|^2)^{1/2}.$$  

This notation is equivalent with $|\xi|$ when $|\xi| \to \infty$, but it avoids the problem of a vanishing denominator when $|\xi| \to 0$.

**Definition 3.37.** Applying this integration by parts $d+1$ times, we obtain an integrand of order $O((y)^{-\infty} \langle \xi \rangle^{-d-1})$, which makes the integral absolutely convergent. This converging integral is taken as the definition for the oscillatory integral $I(\psi)$, and therefore as the definition for the action of the operator $Op_t(a)$ on $\psi \in S(\mathbb{R}^d)$.

This manipulation is equivalent to considering operators with Schwartz kernels $K(x,y)$ given by distributions. Indeed, when the integral over $\xi$ converges absolutely, the Schwartz kernel

$$K(x,y) = \int e^{i\xi (x-y)} a(tx + (1-t)y, \xi) \frac{d\xi}{(2\pi\hbar)^d}$$  

is a well-defined function, the regularity of which depends on the regularity and growth (or decay) of $a(x,\xi)$. We notice that $K(x,y)$ can be easily expressed in terms of the partial Fourier transform of $a$ w.r.t. its momentum variable:

$$(3.45) \quad K(x,y) = [\mathcal{F}_{x \to z} a](tx + (1-t)y, z) \big|_{z = x-y}.$$  

**Lemma 3.38.** If the symbol $a(x,\xi) \in S(\mathbb{R}^{2d})$, then the Schwartz kernel $K(x,y) \in S(\mathbb{R}^{2d})$. Hence, the operator $Op_t(a)$ is continuous as operator $S' \to S$. As a consequence, it is bounded $L^2 \to L^2$.

For symbols $a(x,\xi)$ which are not decaying (or not decaying fast enough), the above integral makes sense as a distribution $K \in S'(\mathbb{R}^{2d})$. This remark allows to quantize rough symbols $a \in S'(\mathbb{R}^{2d})$.

**Proposition 3.39.** If $a \in S'(\mathbb{R}^{2d})$, then for any $t \in [0,1]$ and $\hbar \in (0,1]$ the operator $Op_t(a)$ can be defined as a continuous operator $S(\mathbb{R}^d) \to S'(\mathbb{R}^d)$. 

Proof. The formula (3.45) for the kernel $K(x,y)$ makes sense as a distribution $K \in \mathcal{S}'(\mathbb{R}^{2d}_{x,y})$, and thus defines by duality a continuous operator $\mathcal{S}' \to \mathcal{S}$. □

Remark 3.40. An alternative way to give a meaning to the integral $I(\psi)$ is to insert the factors $e^{-\epsilon(|y|^2 + |\xi|^2)}$, so as to make the integral convergent, and define $I(\psi)$ by its limit $\epsilon \searrow 0$.

Example 3.41. If we take $a(x,\xi) \equiv 1$, we get a representation of the identity, by recovering the fact that the delta function can be written as $\delta(x - y) = \int e^{i\frac{\xi(x-y)}{\hbar}} \frac{d\xi}{(2\pi\hbar)^d}$, equivalently we recover the fact that $\mathcal{F}_\hbar \delta = \frac{1}{(2\pi\hbar)^d}$. More generally, if we take $a(x,\xi) = \xi^\alpha$ for some multi-index $\alpha \in \mathbb{N}^d$, we recover the representation

$$
\left(\frac{\hbar}{i}\right)^{|\alpha|} \delta^\alpha(x-y) = \int e^{i\frac{\xi(x-y)}{\hbar}} \xi^\alpha \frac{d\xi}{(2\pi\hbar)^d},
$$

which shows that $\text{Op}_t(\xi^\alpha) = (\hbar D)^\alpha$.

3.6.2. Symbols with polynomial growth: order functions. Beyond the class $S(\mathbb{R}^{2d})$, we want to extend the quantization map to symbols $a(x,\xi)$ which may grow as $|x|, |\xi| \to \infty$. We’ve seen that the trick of formal integration by parts allows to gain factors $\langle \xi \rangle^{-m}$ or $\langle x \rangle^{-m}$ in the integrals. For this reason, it is necessary to assume that the symbols $a(x,\xi)$ grow at most polynomially.

A convenient way to describe these properties is through the notion of order function.

Definition 3.42. A function $m : \mathbb{R}^{2d} \to \mathbb{R}_+$ is called an order function if there exists $C_0, N_0$ such that

$$
\forall \rho, \rho' \in \mathbb{R}^{2d}, \ m(\rho) \leq C_0 (\rho - \rho')^N m(\rho').
$$

Here we used the Japanese brackets again.

Example 3.43. Typical order functions will be $m(\rho) = \langle \xi \rangle^N$ for some $N$, when we only want growth in momentum, in order to include symbols of the type $|\xi|^2 + V(x)$ with a bounded potential. More generally we can use $m(\rho) = \langle \xi \rangle^N \langle x \rangle^{N_2}$, or $m(\rho) = \langle \rho \rangle^N$ if we want to allow growth both in $\xi$ and $x$.

This order function will allows us to define symbol classes.

Definition 3.44. Let $m(\rho)$ be an order function. Then we define the symbol class $S(\mathbb{R}^{2d}, m) = S(m)$ as follows:

$$
S(m) \overset{\text{def}}{=} \left\{ a = C^\infty(\mathbb{R}^{2d}), \ \forall \alpha \in \mathbb{N}^{2d}, \ \exists C_\alpha > 0, \ \forall \rho \in \mathbb{R}^{2d}, \ |\partial^\alpha_x a(\rho)| \leq C_\alpha m(\rho) \right\}.
$$

Again, we implicitly assume that $a$ can depend on $h \in (0,1]$, such that the above bounds hold uniformly w.r.t. $h$.

Compared with the huge space $\mathcal{S}'(\mathbb{R}^{2d})$, the symbol classes $S(m)$ will allow to control the action of the operators $\text{Op}_h(a)$ on specific Sobolev spaces, a more natural framework for quantum mechanics.

Since we’ll like to compose operators with each other, we are interested in the algebra property of these symbol classes.

Lemma 3.45. For any two order functions $m_1, m_2$, and symbols $a_i \in S(m_i)$, the product symbol $a_1 a_2 \in S(m_1 m_2)$. In particular, the symbol class $S(1)$ is stable by product.
Proof. Obvious consequence of the Leibniz rule.

The Schwartz functions are dense in those spaces, in the following sense:

**Lemma 3.46.** For any \( \epsilon > 0 \), the space \( S(\mathbb{R}^d) \) is dense in \( S(m) \) for the topology of \( S((x, \xi)^{1}\mathbb{R}^*) \).

3.6.3. Action of exponentiated quadratic differentials on \( S(m) \). For symbols \( a \in S(m) \), we want to manipulate operators \( \text{Op}_i(a) \), for instance compose two operators, or compare operators corresponding to different quantizations. As we’ve seen above (see Prop.3.19 and eq.(3.41)), these procedures can be represented by acting on symbols with operators of the type \( e^{i\frac{1}{2}(D,Q^{-1}D)} \) for some symmetric nondegenerate matrix \( Q \) (of dimension \( 2d \) or \( 4d \)). The action of this operator can be defined either on the Fourier side, or on the “direct side” by a convolution operator as in (3.42).

We will therefore study the action of such exponential quadratic derivatives on symbols \( a \in S(m) \) depending on \( x \in \mathbb{R}^n \) (later \( x \rightarrow (x, \xi) \) or \( x \rightarrow (x_0, \xi_0, x_1, \xi_1) \)).

**Proposition 3.47.** Take \( a \in S(m, \mathbb{R}^n) \), and \( Q \) a \( n \times n \) symmetric nondegenerate matrix. Then the distribution \( e^{i\frac{1}{2}(D,Q^{-1}D)}a \) also belongs to \( S(m) \). More precisely, the operator \( e^{i\frac{1}{2}(D,Q^{-1}D)} \) acts continuously \( S(m) \to S(m) \). Moreover, if \( a \) is independent of \( h \), the symbol \( e^{i\frac{1}{2}(D,Q^{-1}D)}a \) admits the asymptotic expansion

\[
e^{i\frac{1}{2}(D,Q^{-1}D)}a \sim \sum_{j \geq 0} \frac{1}{j!} \left( i\frac{1}{2}(D,Q^{-1}D) \right)^j a \quad \text{for the topology of } S(m).
\]

**Proof.** Since the symbol classes \( S(m) \) do not have particularly nice properties w.r.t. the Fourier transform, we will study the operator \( e^{i\frac{1}{2}(D,Q^{-1}D)} \) through its convolution representation:

\[
e^{i\frac{1}{2}(D,Q^{-1}D)}a(x) = C_Q \int_{\mathbb{R}^n} \exp \left( -i \frac{\langle y, Qy \rangle}{2h} \right) a(x + y) \, dy,
\]

with the prefactor \( C_Q = \frac{|\det Q|^{1/2}e^{i\pi \text{sgn} Q/4}}{(2\pi h)^{n/2}} \).

Fixing the point \( x \), we will analyze directly the stationary phase integral

\[
I(x,h) = \int_{\mathbb{R}^n} e^{-i \frac{\langle y, Qy \rangle}{2h}} a(x + y) \, dy.
\]

Since \( a \) is not decaying, we will split the integral between a compactly supported part, containing the stationary point \( y = 0 \), and a noncompact part where the phase oscillates, and where we will be able to gain using integration by parts. Let \( \chi \in C_c^\infty(\mathbb{R}^n) \), \( \chi(y) = 1 \) for \( |y| \leq 1 \), \( \chi(y) = 0 \) for \( |y| \geq 2 \). We split

\[
I(x,h) = I_1(x,h) + I_2(x,h), \quad I_1(x,h) = \int_{\mathbb{R}^n} e^{-i \frac{\langle y, Qy \rangle}{2h}} \chi(y) a(x+y) \, dy, \quad I_2(x,h) = \int_{\mathbb{R}^n} e^{-i \frac{\langle y, Qy \rangle}{2h}} (1 - \chi(y)) a(x+y) \, dy.
\]

The stationary phase estimate (with quadratic phase) applies to \( I_1(h) \), and we get an expansion

\[
I_1(x,h) \sim C_Q^{-1} \sum_{j \geq 0} \frac{1}{j!} \left( i\frac{1}{2}(D_y,Q^{-1}D_y) \right)^j a(x+y) \big|_{y=0}
\]

(observe that the expansion does not depend on \( \chi \), which has a flat germ at \( y = 0 \)). As a function of \( x \), each term in the expansion is bounded above by \( h^{n/2}m(x) \). If we truncate the expansion at the order \( N \), the remainder is bounded by \( 2N + n + 1 \) derivatives of \( a(x+y) \) near \( y = 0 \), so the remainder is bounded
above by \( C_N \hbar^{N+n/2} m(x) \). Hence \( |I_1(x, h)| \leq C \hbar^{n/2} m(x) \). If we differentiate \( I_1 \) w.r.t. \( x \), we get the same expressions, with \( a(x) \) replaced by \( \partial^\alpha a(x) \). As a result, we also get

\[ |\partial^\alpha I_1(x, h)| \leq C_n \hbar^{n/2} m(x). \]

We now want to estimate the nonstationary phase integral \( I_2(x, h) \). We will proceed by integration by parts, using the operator

\[ L = -\frac{\langle Qy, hD_y \rangle}{|Qy|^2} \]

which is well-defined on the support of \( (1 - \chi) \). An important remark is the fact that when \( |y| \to \infty \), the gradient of the norm satisfies \( c|y| \leq |Qy| \leq C|y| \), while the higher derivatives are bounded (or actually vanish from the 3d order). As a result, the \( k \)-th integration by parts of \( I_2(x, h) \) gives

\[ I_2(x, h) = \int_{\mathbb{R}^n} e^{-i\frac{\langle x, Qy \rangle}{\hbar}} t L^k [(1 - \chi(a)) a(x + \bullet)](y) \, dy. \]

The function \( (tL^k) [(1 - \chi(a)) a(x + \bullet)] \) can be estimated by using (3.34):

\[ (tL^k) [(1 - \chi(y)) a(x + y)] \leq C_k \hbar^k \sum_{j=0}^k \frac{|\partial^j (1 - \chi(y)) a(x + y)|}{|y|^{2k-j}} \leq C_k \hbar^k m(x) \leq C_k \hbar^k m(x) |y|^k. \]

Since \( m(x) \leq C \langle y \rangle^{N_0} \) for some \( N_0 \), we see that for \( k \geq N_0 + n + 1 \) the integral becomes absolutely convergent, and satisfies \( |I_2(x, h)| \leq C \hbar^k m(x) \). One way to write this down is \( I_2(x, h) = m(x) O(\hbar^\infty) \).

Again, differentiating \( I_2(x) \) w.r.t. \( x \) amounts to replacing \( a \) with \( \partial^\alpha a \), so we also get

\[ \partial^\alpha I_2(x, h) = m(x) O(\hbar^\infty). \]

Together with (3.46), this shows that \( I(x, h) \in \hbar^{n/2} S(m) \), hence \( e^{i\frac{\hbar}{2}(D,Q^{-1})} a \in S(m) \).

As a first application, we obtain the fact that the symbol class is independent of the chosen quantization.

**Corollary 3.48.** Take \( t, s \in [0, 1] \), and assume \( a_s \in S(m) \) for some order function \( m \). Then the symbol \( a_t \) such that \( Op_t(a_t) = Op_s(a_s) \) also belongs to \( S(m) \).

**Proof.** The explicit formula (3.23) is exactly of the type \( e^{i\frac{\hbar}{2}(D,Q^{-1})} a_s \). □

A more interesting corollary concerns the composition of operators.

**Theorem 3.49.** Take \( a_i \in S(m_i), i = 1, 2 \). Then the symbol \( a_1 \#_h a_2 \in S(m_1 m_2) \). If \( a_i \) are independent of \( h \in (0, 1] \), then the symbol \( a_1 \#_h a_2 \) satisfies the asymptotic expansion

\[ a_1 \#_h a_2 = \sum_{j=0}^{N-1} \frac{(ih/2)^j}{j!} a_1 \omega(D, D) a_2 + O(\hbar^N)_{S(m_1 m_2)}. \]

The treatment of the integral \( I_2(x, h) \) in the proof of Proposition 3.47 has easy, yet important applications.

**Lemma 3.50.** Consider \( a \in C^\infty (\mathbb{R}^{2d}), b \in S(m) \). Then the symbol \( a \#_h b \in S(\mathbb{R}^{2d}) \). More precisely, for any \( \alpha \in \mathbb{N}^{2d} \) and any point \( \rho \notin \text{supp} \ a \) one has the estimate

\[ \partial^\alpha (a \#_h b) (\rho) = O\left( \left( \frac{\hbar}{\text{dist}(\rho, \text{supp} \ a)} \right)^\infty \right). \]
Proof. We use the integral expression

\begin{equation}
(3.48) \quad a_{\#_h} b(\rho) = \iint \frac{dp_0 dp_1}{(2\pi \hbar)^2} \exp \left( -\frac{2i}{\hbar}\omega(\rho_0, \rho_1) \right) a(\rho + \rho_0)b(\rho + \rho_1).
\end{equation}

The integrand is supported in the domain \{(\rho_0, \rho_1) \in (\text{supp } a - \rho) \times (\text{supp } b - \rho)\}. If \rho \notin \text{supp } a this domain does not contain the stationary point (0, 0) but is situated at a distance \(|(\rho_0, \rho_1)| \geq \text{dist } (\rho, \text{supp } a)\) from the stationary point. As a result, we can perform \(k\) integrations by parts in the above integral, leading to factors \(\left(\frac{\hbar}{\|(\rho_0, \rho_1)||}\right)^k\).

If both \(a, b \in C_c^\infty\), the integral is bounded above by \(C h^{-2d} \left(\frac{\hbar}{\|(\rho_0, \rho_1)||}\right)^k \|a\|_{C^k} \|b\|_{C^k}\). Besides, it is compactly supported in \(\rho_0, \rho_1\), and we get the bound

\begin{equation}
(3.49) \quad \|(a_{\#_h} b)(\rho)\| \leq C h^{k-2d} \frac{\|a\|_{C^k} \|b\|_{C^k}}{(\text{dist}(\rho, \text{supp } a) + \text{dist}(\rho, \text{supp } b))^k}, \quad \rho \notin \text{supp } a \cup \text{supp } b.
\end{equation}

In the case \(b \in S(m)\), the integrand is bounded above by

\[ C h^{-2d} m(\rho + \rho_1) \left(\frac{\hbar}{\|(\rho_0, \rho_1)||}\right)^k \leq C h^{k-2d} m(\rho) \left(\frac{\|\rho_1\|}{\|(\rho_0, \rho_1)||}\right)^N \leq C h^{k-2d} m(\rho) \left(\frac{1}{\|(\rho_0, \rho_1)||^{k-N}}\right).\]

The integrand is compactly supported in \(\rho_0\). For \(k > N + 2d\) the integral converges absolutely, and is bounded above by

\[ \|(a_{\#_h} b)(\rho)\| \leq C h^{k-2d} \frac{m(\rho)}{\text{dist}(\rho, \text{supp } a)^{k-N-2d}}.\]

The same estimate holds if we differentiate w.r.t. \(\rho\), which produces the announced estimate. \(\square\)

We notice that, unlike \(a\), the symbol \(c = a_{\#_h} b\) is no more compactly supported. Yet, this symbol is not \(h\)-dependent, and its essential support (the set of points where it is not \(O(h^\infty)\)) is contained in \(\text{supp } a\).

Remark 3.51. A slight modification of the proof shows that if \(a(\rho)\) has an essential support contained in some bounded open set \(\Omega\) (such that \(a(\rho) = O\left(\frac{\hbar}{\text{dist}(\rho, \Omega)}\right)^\infty\) for \(\rho\) outside \(\Omega\)), then the same result applies if we take the symbol \(a_{\#_h} b\) with \(b \in S(m)\).

Let us consider the Moyal product \(a_{\#_h} b\) between two symbols \(a, b \in C_c^\infty(\mathbb{R}^d)\), such that the supports of these two symbols are \(O(1)\), and distant from each other. To fix ideas, these supports have diameters \(\leq 2\), and are centered on points \(z_0, z_1 \in \mathbb{R}^d\) with \(|z_0 - z_1| > 10\). We already know from Corollary 3.36 that \(a_{\#_h} b = O(h^\infty)\). Let us obtain a more precise estimate.

Lemma 3.52. Consider \(a_0, a_1 \in C_c^\infty(\mathbb{R}^d)\), such that \(\text{supp } a_i \subset \{|\rho - z_i| \leq 2\}\), for two points \(z_0, z_1 \in \mathbb{R}^d\) at distance \(|z_0 - z_1| \geq 10\).

\[ |\partial^\alpha (a_0_{\#_h} a_1)(\rho)| \leq C_{\alpha} h^{k-2d} \frac{1}{(|z_1 - z_0|^2 + |\rho - \frac{z_1 + z_0}{2}|^2)^{k/2}}.\]

Proof. We analyze the integral (3.48). The integrand is supported on points \((\rho_0, \rho_1) \in \{|z_0 - \rho| \leq 2, |z_0 - \rho| \leq 2\}\). Hence, using the estimate on the operator \(tL^k\) as in (3.47), one can easily show that

\[ |a_0_{\#_h} a_1(\rho)| \leq C h^{k-2d} \frac{1}{(|z_0 - \rho|^2 + |z_1 - \rho|^2)^{k/2}} \leq C' h^{k-2d} \frac{1}{(|z_1 - z_0|^2 + |\rho - \frac{z_1 + z_0}{2}|^2)^{k/2}}.\]
As before, the same type of estimate holds for derivatives w.r.t. \( \rho \). \( \square \)

3.7. **Action of pseudodifferential operators on \( L^2(\mathbb{R}^d) \).** So far we have considered the action of operators \( \text{Op}_\hbar(a) \) on \( \psi \in \mathcal{S}(\mathbb{R}^d) \). However, in quantum mechanics the natural functional space is the Hilbert space \( L^2(\mathbb{R}^d) \), or its Sobolev descendents \( H^s(\mathbb{R}^d) \).

3.7.1. **Action of Schwartz symbols on \( L^2 \).** Let us start with nice symbols \( a \in \mathcal{S}(\mathbb{R}^{2d}) \). The Fourier expression (3.21) for \( \text{Op}_t(a) \) is very convenient. Indeed, it is straightforward to check that the phase space translation operators \( \text{Op}_W^\hbar(e_{1_0}) \) are all unitary on \( L^2 \). Using the triangular inequality we get:

**Lemma 3.53.** Assume \( a \in \mathcal{S}(\mathbb{R}^{2d}) \). Then for any \( t \in [0,1] \) the operator \( \text{Op}_t(a) \) is bounded on \( L^2 \):

\[
\| \text{Op}_t(a) \|_{L^2 \to L^2} \leq \frac{1}{(2\pi)^d} \| \hat{a} \|_{L^1(\mathbb{R}^{2d})} \leq C_d \sum_{|\alpha| \leq 2d+1} \| \partial^\alpha a \|_{L^1(\mathbb{R}^{2d})}.
\]

An alternative proof of the boundedness of \( \text{Op}_\hbar(a) \) uses the explicit expression of the Schwartz kernel \( K_a(x,y) \), which allows to use Schur’s inequality. We will see below that we can get a sharper estimate on this bound in terms of the symbol \( a \).

**Lemma 3.54.** (Schur’s inequality) Assume that the Schwartz kernel \( K(x,y) \) of an operator \( A \) on \( \mathcal{S}(\mathbb{R}^d) \) satisfies

\[
\sup_x \int dy |K(x,y)| < C_1, \quad \sup_y \int dx |K(x,y)| < C_2.
\]

Then \( A \) is bounded on \( L^2(\mathbb{R}^d) \), with the bound \( \| A \|_{L^2 \to L^2} \leq \sqrt{C_1C_2} \).

We will now attack a less elementary task, which is to show that for any symbol \( a \) in the class \( S(1) \) (all derivatives are uniformly bounded on \( \mathbb{R}^d \)), the operators \( \text{Op}_t(a) \) are bounded on \( L^2 \) (uniformly w.r.t. \( \hbar \)). We already know that this is the case for symbols of the form \( f(x) \) or \( g(\xi) \), since these operators act by multiplication on \( L^2_\mathbb{R} \), resp. on \( L^2_\mathbb{C} \). The proof for a general symbols \( a \in S(1) \) is much less straightforward. The idea is to split the symbol \( a \) into countably many symbols \( a_n \), each of them being supported in an \( O(1) \) neighbourhood of the lattice point \( n \in \mathbb{Z}^{2d} \). We know from Lemma that each \( \text{Op}_\hbar(a_n) \) is bounded, and we know that two operators \( \text{Op}_\hbar(a_n) \), \( \text{Op}_\hbar(a_{n'}) \) are “quasi-orthogonal” to each other if \( n, n' \) are distant from each other (cf. Lemma 3.52). The idea is thus to show that the action of \( \text{Op}_\hbar(a) \) can be seen as the combination of the actions of the \( \text{Op}_\hbar(a_n) \), each one acting on (quasi-)orthogonal subspaces of \( L^2 \).

To make this “quasi-orthogonality” into an effective bound, we use the **Cotlar-Stein Theorem**, an abstract operator theoretic result.

**Theorem 3.55.** (Cotlar-Stein Theorem) Let \( (A_j)_{j \geq 1} \) be a family of bounded operators on some Hilbert space \( \mathcal{H} \), and assume that the following bounds hold:

\[
(3.50) \quad \sup_j \sum_k \| A_j^* A_k \|^{1/2} \leq C, \quad \sup_j \sum_k \| A_j A_k^* \|^{1/2} \leq C.
\]

Then the series \( \sum_j A_j \) converges (in the strong operator topology) to an operator \( A \), which satisfies \( \| A \| \leq C \).
Notice that the sum $\sum_j A_j$ certainly does not converge absolutely, since the norms $\|A_k\|$ are not supposed to decay when $k \to \infty$, but they are uniformly bounded.

**Proof.** We first truncate the sum to $A = A^{(J)} = \sum_{j=1}^{J} A_j$, so that all is well-defined. $A$ is a bounded operator, so $A^*A$ is a positive selfadjoint operator, which satisfies

$$\| (A^*A)^m \| = \| A^*A \|^m = \| A \|^{2m}. $$

We want to estimate the norm of $(A^*A)^m$ in a clever way. From the decomposition of $A$, we write

$$(A^*A)^m = \sum_{j_1,j_2,j_2m} A_{j_1}^* A_{j_2} A_{j_3}^* \cdots A_{j_2m} \stackrel{\text{def}}{=} \sum_{j_1,j_2,j_2m} a_{j_1 \cdots j_2n}.$$ 

The trick is to estimate the norm of $a_{j_1 \cdots j_2n}$ in two ways:

$$\| a_{j_1 \cdots j_2n} \| \leq \| A_{j_1}^* A_{j_2} \| \| A_{j_3}^* A_{j_4} \| \cdots \| A_{j_{2m-1}}^* A_{j_{2m}} \|$$

$$\| a_{j_1 \cdots j_2n} \| \leq \| A_{j_1}^* \| \| A_{j_2}^* A_{j_3} \| \cdots \| A_{j_{2m}} \|. $$

Taking the geometric mean of these two bounds (and noticing that $\| A_j \| \leq C$ from our assumptions), we get

$$\| a_{j_1 \cdots j_2n} \| \leq C \left( \| A_{j_1}^* A_{j_2} \| \right)^{1/2} \left( \| A_{j_2}^* A_{j_3} \| \right)^{1/2} \cdots \left( \| A_{j_{2m-1}}^* A_{j_{2m}} \| \right)^{1/2}. $$

Through the triangular inequality, this gives the following bound for $\| (A^*A)^m \|$:

$$\| (A^*A)^m \| \leq C \sum_{j_1,j_2,j_2m} \| A_{j_1}^* A_{j_2} \|^{1/2} \| A_{j_2}^* A_{j_3} \|^{1/2} \cdots \| A_{j_{2m}} \|^{1/2}. $$

If we first sum over $j_1$ using the assumption, producing a factor $C$. Then we sum over $j_2$, etc. In the end we sum over $j_{2n}$, which produces a factor $J$. This gives $\| (A^*A)^m \| \leq C J^{2m-1} J$, and therefore $\| A \| \leq C J^{1/2m}$. Since this estimate holds for any $m \geq 1$, we get

$$\| A^{(J)} \| \leq C.$$ 

Let us now prove the strong convergence when $J \to \infty$. Take $\psi \in H$, and consider $\varphi = A_{k_0}^* \psi$. Then we may write formally

$$\sum_{j \geq 1} A_j \varphi = \sum_{j \geq 1} A_j A_{k_0}^* \psi,$$

and this series converges absolutely, since

$$\sum_j \| A_j A_{k_0}^* \psi \| \leq \sum_j \| A_j A_{k_0}^* \| \| \psi \|$$

$$\leq \sum_j \| A_j A_{k_0}^* \|^{1/2} \| A_j A_{k_0}^* \|^{1/2} \| \psi \|$$

$$\leq \sum_{j,j'} \| A_j A_{k_0}^* \|^{1/2} \| A_{j'} A_{k_0}^* \|^{1/2} \| \psi \|$$

$$\leq C^2 \| \psi \|.$$ 

Hence the limit $A \varphi = \lim_{J \to \infty} A^{(J)} \varphi$ converges for any $\varphi \in \text{span} \{ A_{k}^* (H), \ k \geq 1 \}$. On the other hand, we have proved that $\| A^{(J)} \| \leq C$ uniformly for all $J$. We thus deduce that $\| A \varphi \| \leq C \| \varphi \|$ for any
\( \varphi \in \text{span}\{A_k^*(\mathcal{H}),\ k \geq 1\} \). It is then possible to extend \( A \) to any \( \varphi \) in the closure of this subspace, keeping the same bound \( \|A\varphi\| \leq C\|\varphi\| \). What is the orthogonal complement of that closure? It is the subspace \( \bigcap_k \ker A_k \). For states in this subspace, we naturally have \( A\varphi = 0 \). Finally, we have defined \( A\varphi \) for all states \( \varphi \in \mathcal{H} \), and showed that it satisfies the announced bound. \( \square \)

With this Cotlar-Stein theorem, we are now equipped to prove the \( L^2 \) continuity of pseudodifferential operators with symbols in \( S(1) \), namely the following Theorem.

**Theorem 3.56.** (Calderon-Vaillancourt Theorem) Let \( a = a(h) \in S(1, \mathbb{R}^{2d}) \). Then the operator \( \text{Op}_h^W(a) \) can be extended as a bounded operator on \( L^2(\mathbb{R}^d) \), with a bound uniform w.r.t. \( h \in (0, 1] \).

More precisely, there exists constants \( M > 0 \) and \( C_d > 0 \) such that

\[
(3.51) \quad \left\| \text{Op}_h^W(a) \right\|_{L^2 \to L^2} \leq C_d \sum_{|\alpha| \leq 6d+2} h^{\alpha/2} \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^{2d})},
\]

The same estimate holds if we replace the Weyl quantization by any \( t \)-quantization.

**Proof.** As mentioned above, the proof proceeds by splitting the symbol into infinitely many compactly supported operators. We start by building up a smooth partition of unity on \( \mathbb{R}^{2d} \). Consider a cutoff function \( \tilde{\chi} \in C_c^\infty(\mathbb{R}^{2d}) \), supported in \( \{|\rho| \leq R_d\} \), strictly positive in \( \{|\rho| \leq R_d/2\} \). If \( R_d \) is sufficiently large (in practice, \( R_d \geq \sqrt{d} \) should work), then the function

\[
S(\rho) = \sum_{n \in \mathbb{Z}^{2d}} \tilde{\chi}(\rho - n)
\]

is everywhere positive. It is also periodic. Hence, if we take \( \chi(\rho) = \frac{\tilde{\chi}(\rho)}{S(\rho)} \), we get a smooth partition of unity:

\[
\sum_{n \in \mathbb{Z}^{2d}} \chi(\rho - n) \equiv 1.
\]

We then set \( a_n(\rho) = a(\rho) \chi(\rho - n) \), which is a symbol compactly supported in the ball \( B(n, R_d) \). Notice that the \( S(1) \) seminorms of \( a_n \) are controlled by those of \( \chi \) and \( a \). By linearity, we have

\[
\text{Op}_h(a) = \sum_{n \in \mathbb{Z}^{2d}} \text{Op}_h(a_n).
\]

A priori, the sum on the RHS is well-defined as an operator on \( S(\mathbb{R}^d) \), but it is not clear how it acts on \( L^2 \). If we call \( A_n = \text{Op}_h(a_n) \) the Cotlar-Stein theorem suggests to study the norms of the operators \( \text{Op}_h(a_n)^* \text{Op}_h(a_{n'}) \) and \( \text{Op}_h(a_{n'}) \text{Op}_h(a_n)^* = \text{Op}_h(a_{n'}#_h a_n) \).

For \( |n-n'| \leq 10R_d \) we apply the bounds of Lemma 3.33 for the seminorms of \( a_n#_h a_{n'} \). For \( \rho \) close to \( n \) we have

\[
|\partial^\alpha (a_n#_h a_{n'}) (\rho)| \leq C_{\gamma, \alpha} \left\| \langle D \rangle^{\alpha|+2d+1} a_n \right\|_{L^\infty} \left\| \langle D \rangle^{\alpha|+2d+1} a_{n'} \right\|_{L^\infty},
\]

where we used the fact that the symbols \( a_n \) are compactly supported near \( n \). For \( \rho \) away from the support of \( a_n \), the Lemma 3.50 implies the bound

\[
|\partial^\alpha (a_n#_h a_{n'}) (\rho)| \leq C h^{k-2d} \frac{\|a_n\|_{C_{k+|\alpha|}} \|a_{n'}\|_{C_{k+|\alpha|}}}{(\text{dist}(\rho, \text{supp } a_n) + \text{dist}(\rho, \text{supp } a_{n'}))^k}.
\]
Taking $k = 2d + 1$ to have integrability, we may apply the Lemma 3.53 and get the estimate
\begin{equation}
\|A_n^* A_n\|_{L^2 \to L^2} \leq C_d \|a_n\|_{C^{d+2}} \|a_n'\|_{C^{d+2}} \leq C_d \|a\|_{C^{d+2}}^2.
\end{equation}

When $|n - n'| > 10R_d$, the Lemma 3.52 shows that, these symbols satisfy bounds of the type
\begin{equation}
|\partial^n (\bar{a}_n \# a_{n'}) (\rho)| \leq C_k h^{k-2d} \frac{\|a_n\|_{C^{k+\gamma}} \|a_n'\|_{C^{k+\gamma}}}{(|n - n'|)^k} (\frac{|n - n'|^2 + |\rho - \frac{n + n'}{2}|^2}{2})^{k/2}
\end{equation}
(where the constants implicitly depend on the cutoff $\chi$). If we take $k > 2d$ the RHS is integrable, so we may invoke Lemma 3.53 to get the $L^2$ bounds
\begin{equation}
\|A_n^* A_n\|_{L^2 \to L^2} \leq C_k h^{k-2d} \frac{\|a_n\|_{C^{k+2d+1}} \|a_n'\|_{C^{k+2d+1}}}{(|n - n'|)^k} \leq C_{k, \chi} h^{k-2d} \frac{\|a\|_{C^{k+2d+1}}^2}{(|n - n'|)^k}, \quad k \geq 2d + 1.
\end{equation}
The same bound holds for the norm $\|A_n A_n^*\|$. By taking $k \geq 4d + 1$, we see that the expressions $\sum_{n'} \|A_n A_n^*\|^{1/2}$, $\sum_{n'} \|A_n A_n^*\|^{1/2}$ converge, so we may recover the assumption of the Cotlar-Stein Theorem:
\begin{equation}
\sup_n \sum_{n'} \|A_n A_n^*\|^{1/2} \leq C_{d, \chi} \|a\|_{C^{6d+2}}, \quad \sup_n \sum_{n'} \|A_n A_n^*\|^{1/2} \leq C_{d, \chi} \|a\|_{C^{6d+2}}.
\end{equation}
These estimates allow to apply the Cotlar-Stein Theorem, which shows that $\text{Op}_h(a)$ is well-defined as a bounded operator on $L^2$, with a norm
\begin{equation}
\|\text{Op}_h(a)\|_{L^2 \to L^2} \leq C_d \|a\|_{C^{6d+2}} = C_d \sum_{|\alpha| \leq 6d+2} \|\partial^n a\|_{L^\infty}, \quad \text{uniformly for } h \in (0, 1].
\end{equation}
To improve the bound, we proceed by a simple scaling argument. Namely, and the unitary rescaling operator
\begin{equation}
U_h \psi(x) \overset{\text{def}}{=} h^{d/2} \psi(h^{1/2} x),
\end{equation}
we obtain the identity
\begin{equation}
U_h \text{Op}_h W(a) \psi = \text{Op}_1 W(\tilde{a}) U_h \psi,
\end{equation}
where $\tilde{a}$ is the rescaled symbol $\tilde{a}(\rho) = a(h^{1/2} \rho)$. Indeed:
\begin{align*}
\hbar^{d/2} \left[ \text{Op}_h W(a) \psi \right] (h^{1/2} x) &= h^{d/2} \int \frac{dy d\xi}{(2\pi\hbar)^d} e^{\frac{i}{\hbar} \xi(h^{1/2} x - y)} a \left( \frac{h^{1/2} x + y}{2}, \frac{\xi}{2} \right) \psi(y) \\
&= \int \frac{dY d\Xi}{(2\pi)^d} e^{\frac{i}{\hbar} \Xi(h^{1/2} x - h^{1/2} Y)} a \left( \frac{h^{1/2} x + h^{1/2} Y}{2}, \frac{h^{1/2} \Xi}{2} \right) h^{d/2} \psi(h^{1/2} Y) \\
&= \int \frac{dY d\Xi}{(2\pi)^d} e^{\frac{i}{\hbar} \Xi(x-Y)} \tilde{a} \left( \frac{x + Y}{2}, \frac{\Xi}{2} \right) h^{d/2} \psi(h^{1/2} Y) \\
&= \left[ \text{Op}_1 W(\tilde{a}) U_h \psi \right] (x).
\end{align*}
This shows that the operators $\text{Op}_h W(a)$, $\text{Op}_1 W(\tilde{a})$ are unitarily conjugate, thus they have the same $L^2 \to L^2$ norm. Now, if we can apply the estimate (3.54) to $\text{Op}_1 W(\tilde{a})$, and notice that $\|\partial^n \tilde{a}\|_{L^\infty} = h^{n/2} \|\partial^n a\|_{L^\infty}$, we obtain the announced Bound.
Remark 3.57. We will show below that the bound (3.51) can be slightly improved, namely the $\mathcal{O}(1)$ term is actually $\|a\|_{L^\infty}$ (with no extra constant).

The Calderon-Vaillancourt theorem is very important. It allows to transform remainder terms, expressed in the topology of $S(1)$, into remainder terms in the topology of operators on $L^2$, which is more handy (and natural) when we study spectral questions or time evolution on $L^2$. A first example, direct corollary of the composition theorem 3.49:

**Corollary 3.58.** *(Pseudodifferential calculus on $L^2$)* Take two symbols $a, b \in S(1)$. We already know that $a \# b \in S(1)$. The asymptotic expansion of Theorem 3.49 translates into

$$\text{Op}_h^W(a) \circ \text{Op}_h^W(b) = \sum_{j=0}^{N-1} \frac{(ih/2)^j}{j!} \text{Op}_h^W\left(a \left(\omega(\overrightarrow{D}, \overrightarrow{D})\right)^j b\right) + \mathcal{O}(h^N)_{L^2 \to L^2},$$

where the implied constant depends on a certain number of derivatives of $a, b$.

At the first order level,

$$\text{Op}_h^W(a) \circ \text{Op}_h^W(b) = \text{Op}_h^W(ab) - \frac{ih}{2} \text{Op}_h^W\{a, b\} + \mathcal{O}(h^2)_{L^2 \to L^2},$$

This is a manifestation of the quantum-classical correspondence, now in the $L^2$ framework.

For two symbols $a, b \in S(1)$ with disjoint supports, the above expansion shows that

$$\text{Op}_h^W(a) \circ \text{Op}_h^W(b) = \mathcal{O}(h^\infty)_{L^2 \to L^2}.$$  

3.7.2. *From $L^2$ properties of the operator to those of its symbol.* The C-V theorem has a sort of “inverse”, namely we can deduce properties of the symbol $a \in S'(\mathbb{R}^{2d})$ from the $L^2$ properties of the operator $\text{Op}_h(a)$. We first state a result concerning the case $h = 1$.

**Proposition 3.59.** Let $a \in S'(\mathbb{R}^{2d})$. We assume that the operators $\text{Op}(\partial^\alpha a)$ are bounded $L^2 \to L^2$ for all derivatives of order $|\alpha| \leq 2d + 1$. Then $a \in L^\infty(\mathbb{R}^{2d})$, and we have the estimate

$$\|a\|_{L^\infty} \leq C_d \sum_{|\alpha| \leq 2d+1} \|\text{Op}_1(\partial^\alpha a)\|_{L^2 \to L^2}.$$  

Using the $h^{1/2}$-rescaling which connects $\text{Op}_h(a)$ with $\text{Op}_1(\tilde{a})$, we obtain a more precise result in case of the $h$-quantization:

$$\|a\|_{L^\infty} \leq C_d \sum_{|\alpha| \leq 2d+1} h^{|\alpha|/2} \|\text{Op}_h(\partial^\alpha a)\|_{L^2 \to L^2}.  \tag{3.55}$$

From this estimate we can straightforwardly deduce Beals’s Lemma, which allows to characterize operators with symbols in $S(1)$. This characterization uses the commutators of $\text{Op}_h(a)$ with the quantizations of linear symbols $\ell(x, \xi) = \xi_0 \cdot x - x_0 \cdot \xi$. The adjoint action of $\text{Op}_h(\ell)$ on some operator $A$ is defined by the commutator

$$\text{ad}_{\text{Op}_h(\ell)} A = [\text{Op}_h(\ell), A].$$

**Theorem 3.60.** *(Beals’s Theorem)* Let $a \in S'(\mathbb{R}^{2d})$, possibly depending on $h \in (0, 1]$, and take $A = \text{Op}_h(a)$. Then the two followings statements are equivalent:
1) \( a \in S(1) \).

2) for every \( N \geq 0 \) and every sequence \( \ell_1, \ldots, \ell_N \) of linear symbols, the operator \( \text{ad}_{\text{Op}_h(\ell_N)} \circ \cdots \circ \text{ad}_{\text{Op}_h(\ell_1)} A \) is bounded on \( L^2 \), with norm

\[
(3.56) \quad \left\| \text{ad}_{\text{Op}_h(\ell_N)} \circ \cdots \circ \text{ad}_{\text{Op}_h(\ell_1)} A \right\|_{L^2 \to L^2} = O_N(h^N).
\]

**Proof.** We simply notice that \( \text{ad}_{\text{Op}_h(\ell_1)} A \) involves derivatives of \( a(\rho) \). The commutation with linear symbols is covariant with quantization, in the sense that the first-order expansion of the Moyal product is exact:

\[
[\text{Op}_h(\ell), \text{Op}_h(a)] = -i\hbar \text{Op}_h(\{\ell, a\})
\]

so the assumption implies that \( \left\| \text{Op}_h(\partial a) \right\| = O(1) \). Proceeding by iterations, we find that \( \left\| \text{Op}_h(\partial^\alpha a) \right\| = O(\alpha) \) for all \( \alpha \in \mathbb{N}^{2d} \). Injecting these estimates in (3.55) we get, for any \( \beta \in \mathbb{N}^{2d} \),

\[
\left\| \partial^\beta a \right\|_{L^\infty} \leq C_d \sum_{|\alpha| \leq 2d+1} h^{|\alpha|/2} \left\| \text{Op}_h(\partial^{\beta+\alpha} a) \right\|_{L^2 \to L^2} = O(1),
\]

which shows that \( a \in S(1) \). \( \square \)

### 3.8. Compact, Hilbert-Schmidt, Trace class pseudodifferential operators.

We now study in more detail the pseudodifferential \( \text{Op}_h(a) \) on \( L^2(\mathbb{R}^d) \), with a view on their spectral properties.

One of our objectives is to find criteria for our operators to have discrete spectra. For this we will use a caracterization of compact operators, since one way to prove discreteness of the spectrum of a symmetric operator \( A \) is to show that its resolvent \( (A - i)^{-1} \) is compact. In a second step, we will be interested in counting the eigenvalues of \( A \), and for this we will use the functional calculus of pseudodifferential operators.

We recall a few definitions.

**Definition 3.61.** An operator \( A : L^2 \to L^2 \) is compact if it maps any bounded subset of \( L^2 \) into a precompact set of \( L^2 \). Equivalently, for any sequence \( (\psi_j) \) bounded in \( L^2 \), one can extract from the sequence \( (A\psi_j) \) a converging subsequence.

**Proposition 3.62.** The spectrum of a compact operator \( A \) is made of eigenvalues \( \mu_i \neq 0 \) with finite multiplicities, which only possible accumulation point being the origin.

If \( A \) is compact, then \( A^*A \) and \( AA^* \) are compact and selfadjoint, their nonzero eigenvalues can be called \( (s_j^2)_{j \geq 0} \) (in decreasing order). The \( (s_j) \) are called the *singular values* of \( A \).

**Definition 3.63.** \( A \) is Hilbert-Schmidt if \( \|A\|_{HS}^2 = \sum_j s_j^2 < 0 \). \( A \) is trace-class if \( \|A\|_{tr}^2 = \sum_j s_j < \infty \).

If \( A \) is trace-class, it admits a linear functional call its trace, defined by \( \text{tr}A \overset{\text{def}}{=} \sum_j \langle e_j, Ae_j \rangle \), for any orthonormal basis \( (e_j) \). On has \( |\text{tr}A| \leq \|A\|_{tr} \).

Notice that \( A \) trace class \( \Rightarrow A \) Hilbert-Schmidt. Trace-class and HS form ideals of the space of bounded operators.
One also has
\[(3.57) \quad \|A\|_{tr} = \sup_{(e_j),(f_j)} \sum_j \langle e_j, Af_j \rangle,\]
where the supremum is taken over all pairs of ONB.

If $A, B$ are HS, then $AB$ is trace-class, and $\|BA\|_{tr} \leq \|B\|_{HS} \|A\|_{HS}$. The trace enjoys the cyclic property $\text{tr}(AB) = \text{tr}(BA)$ (also true of $A$ is trace-class and $B$ bounded).

**Proposition 3.64.** On $L^2(\mathbb{R}^d)$, an operator $A$ with Schwartz kernel $K(x,y)$ is HS iff $K \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$, and
\[(3.58) \quad \|A\|_{HS} = \|K\|_{L^2}.\]

**Corollary 3.65.** Take $a \in \mathcal{S}'(\mathbb{R}^{2d})$. Then $\text{Op}_W^W(a)$ is HS iff $a \in L^2(\mathbb{R}^{2d})$, and one has the identity
\[
\|\text{Op}_W^W(a)\|_{HS}^2 = \int |a(x,\xi)|^2 \frac{dx\,d\xi}{(2\pi\hbar)^d} = \frac{1}{(2\pi\hbar)^d} \|a\|_{L^2}^2.
\]

**Proof.** Start from the characterization (3.57), and recall the relationship between kernel and symbol:
\[
K(x,y) = \int a \left( \frac{x+y}{2}, \xi \right) e^{i \frac{x-y}{\hbar} \cdot \xi} \frac{d\xi}{(2\pi\hbar)^d} = \frac{1}{(2\pi\hbar)^{d/2}} \left( F_{\hbar\xi \rightarrow \bullet} a \right) \left( \frac{x+y}{2}, x-y \right).
\]
Using the fact that the change of variables $(x,y) \mapsto \left( \frac{x+y}{2}, x-y \right)$ has Jacobian unity, we get that
\[
\int |K(x,y)|^2 dx\,dy = \int \left| \left( F_{\hbar\xi \rightarrow \bullet} a \right) \left( \frac{x+y}{2}, x-y \right) \right|^2 dx\,dy.
\]
This leads to a first criterium to ensure compactness of PDOs (since HS operators are automatically compact).

**Corollary 3.66.** If $a \in S(m)$ with and order function $m(\rho)$ square-integrable, then $\text{Op}_W^W(a)$ is HS, hence compact.

Let now give a more general (and less quantitative) criterium for a PDO to be compact.

**Theorem 3.67.** If $a \in S(m)$ with an order function $m(\rho) \overset{\rho \to \infty}{\to} 0$, then $\text{Op}_W^W(a)$ is a compact operator.

**Proof.** We use the decomposition of $a(\rho)$ used in the proof of the Calderon-Vaillancourt Theorem, namely we split it into compactly supported symbols:
\[
a = \sum_{n \in \mathbb{Z}^{2d}} a_n.
\]
We know that each $\text{Op}_\hbar(a)$ is compact, therefore $A_M \overset{\text{def}}{=} \sum_{|n| \leq M} A_n$ is compact for any $M > 0$. The ideal of compact operators is closed w.r.t. the operator norm, so it is enough to show that
\[(3.59) \quad \|A - A_M\|_{L^2 \rightarrow L^2} \overset{M \to \infty}{\longrightarrow} 0.
\]
To prove this limit we use the Cotlar-Stein Lemma, applied to the operator
\[
B_M = A - A_M = \sum_{|n| > M} A_n.
\]
Indeed, for \( a \in S(m) \) the estimates (3.52,3.53) are easily modified into
\[
\|A_n^*A_n\|_{L^2 \to L^2} \leq C_{a,d}m(n)m(n') \quad |n - n'| \leq 10R_d,
\]
\[
|n - n'| > 10R_d, \quad k \geq 2d + 1.
\]

Hence, for any \( n \in \mathbb{Z}^{2d} \) we may write
\[
\sum_{|n'| > M} \|A_n^*A_n\|^{1/2} \leq C_{a,d} \sum_{|n'| > M} \frac{\sqrt{m(n)m(n')}}{|n - n'|^{k/2}} \leq C_{a,d}m(n),
\]
where we used the defining property of an order function, and took \( k \) large enough to have the sum converge. As a result we get
\[
\sup_{|n| > M} \sum_{|n'| > M} \|A_n^*A_n\|^{1/2} \leq C_{a,d} \sup_{|n| > M} m(n) M \to \infty \to 0,
\]
The same convergence holds for the sums \( \sum_{n'} \|A_n^*A_n\|^{1/2} \). Applying the Cotlar-Stein Theorem shows the limit (3.58), and proves the theorem.

### 3.8.1. Criteria for trace class operators.

Let us now investigate criterions for a PDO to be trace-class. Let us first consider the case of an operator \( A \) with Schwartz kernel \( K(x,y) \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d) \). If we consider a cutoff function \( \chi \in C_c^\infty(\mathbb{R}^d) \) such that \( \chi(x)\chi(y) \equiv 1 \) on the support of \( K \), we have
\[
K(x,y) = \chi(x)\chi(y)K(x,y) = \int \hat{K}(\xi,\eta)e^{i(\xi \cdot x + \eta \cdot y)}\chi(x)\chi(y) \frac{d\xi d\eta}{(2\pi)^d}, \quad \text{with the notation } \hat{K} = \mathcal{F}_1K.
\]

For each \( \xi, \eta \), the kernel \( e^{i(\xi \cdot x + \eta \cdot y)}\chi(x)\chi(y) \) represents the rank-1 operator \( \psi \mapsto \chi_\xi(\chi_{-\eta}, \psi) \), where \( \chi_\xi \) is the state with wavefunction \( e^{i\xi \cdot x}\chi(x) \). The formula (3.56) shows that this operator has trace norm \( ||\chi||_{L^2}^2 \), and a trace \( \text{tr} (\chi_\xi(\chi_{-\eta}, \cdot)) = \int e^{i(\xi + \eta \cdot) \cdot x}\chi(x)^2 dx \), so we get
\[
\|A\|_{tr} \leq \|\chi\|_{L^2}^2 \int \left| \int \hat{K}(\xi,\eta) \frac{d\xi d\eta}{(2\pi)^d} \right| = \|\chi\|_{L^2}^2 \|\hat{K}\|_{L^1},
\]
\[
\text{tr}A = \int \int \hat{K}(\xi,\eta) e^{i(\xi + \eta \cdot) \cdot x}\chi(x)^2 \frac{d\xi d\eta}{(2\pi)^d} = \int K(x,x) dx.
\]

For a more general kernel we may split \( K \) using a partition of unity of \( \mathbb{R}^d \):
\[
K(x,y) = \sum_{n,n' \in \mathbb{Z}^d} K_{n,n'}(x,y), \quad K_{n,n'}(x,y) = K(x,y)\chi(x - n)\chi(y - n').
\]

By linearity, if the norms \( \|\hat{K}_{n,n'}\|_{L^1} \) decay sufficiently fast so that
\[
\sum_{n,n'} \|\hat{K}_{n,n'}\|_{L^1} < \infty,
\]
then we deduce that \( A \) is trace class, with norm
\[
\|A\|_{tr} \leq C_d \sum_{n,n'} \|\hat{K}_{n,n'}\|_{L^1} < \infty.
\]
Now by the standard Fourier transform estimate we have
\[ \| \hat{K}_{n,n'} \|_{L^1} \leq C_d \sum_{|\alpha| \leq 2d+1} \| \partial^\alpha K_{n,n'} \|_{L^1}. \]

Taking into account the partition of unity (3.61) and the fact that the derivatives of \( \chi \) are bounded, we find that
\[ \|A\|_{tr} \leq \sum_{|\alpha| \leq 2d+1} \sum_{n,n'} \| \partial^\alpha K_{n,n'} \|_{L^1} \leq C_{d,\chi} \sum_{|\alpha| \leq 2d+1} \| \partial^\alpha x,y K \|_{L^1(dx dy)}. \]

If the RHS is finite, the function \( x \mapsto K(x,x) \) is then automatically continuous, bounded and integrable, and we have by linearity
\[ \text{tr} A = \int K(x,x) \, dx. \]

We have showed the following

**Proposition 3.68.** Assume that Schwartz kernel \( K(x,y) \) of an operator \( A : L^2 \to L^2 \) satisfies
\[ \sum_{|\alpha| \leq 2d+1} \| \partial^\alpha x,y K \|_{L^1(dx dy)} < \infty. \]

Then this operator is trace class, and its norm is bounded by
\[ (3.63) \quad \|A\|_{tr} \leq C_d \sum_{|\alpha| \leq 2d+1} \| \partial^\alpha x,y K \|_{L^1(dx dy)}. \]

The kernel on the diagonal \( K(x,x) \) is integrable, and the trace of \( A \) is given by the integral
\[ \text{tr}(A) = \int K(x,x) \, dx. \]

Let us now try to relate the trace class property of a PDO with those of its symbol. We start with a simple consequence of the above Proposition:

**Corollary 3.69.** If \( a \in S(\mathbb{R}^{2d}) \), the Schwartz kernel \( K_1(x,y) \) of \( \text{Op}_1(a) \) is also in \( S(\mathbb{R}^d \times \mathbb{R}^d) \), so the above criteria hold. We have in general
\[ \text{tr} \text{Op}_1(a) = \frac{1}{(2\pi \hbar)^d} \int a(x,\xi) \, dx \, d\xi. \]

Let us now consider more general symbols and look for a criterion for being trace-class. For simplicity we start using the right quantization \( \text{Op}_1^R(a) \), which we may write as
\[ \text{Op}_1^R(a) \psi = \int e^{i\xi \cdot x} a(x,\xi) F_{\eta} \psi(\xi) \frac{d\xi}{(2\pi)^d}. \]

Since the Fourier transform is unitary, we need to study the operator with kernel \( K(x,\xi) = e^{i\xi \cdot x} a(x,\xi) \) (viewing \( \xi \) as the initial variable).

It is not a clever idea to use directly the estimate (3.62) for this kernel, since we get a (bad) factor \( \xi \) each time we differentiate it w.r.t. \( x \), and vice-versa. Let us rather study the cutoff kernel
\[ K_{n,n'}(x,\xi) = \chi(x-n)\chi(\xi-n')e^{i\xi \cdot x} a(x,\xi). \]
We want to rewrite the phase \( \xi \cdot x = (\xi - \mathbf{n}') \cdot (x - \mathbf{n}) + \xi \cdot \mathbf{n} + \mathbf{n}' \cdot x - \mathbf{n} \cdot \mathbf{n}' \), so that the derivatives of the first term remains uniformly bounded. The terms \( \xi \cdot \mathbf{n} + \mathbf{n}' \cdot x \) produce a shift in the Fourier transform, and we get

\[
\mathcal{F}_1 K_{n,n'}(x^*, \xi^*) = e^{-in' \cdot x} \mathcal{F}_1 \left( \chi(x - \mathbf{n})\chi(\xi - \mathbf{n}')e^{i(\xi - \mathbf{n}) \cdot (x - \mathbf{n})} a(x, \xi) \right) (x^* - \mathbf{n}', \xi^* - \mathbf{n}).
\]

Let us call the function \( a_{n,n'}(x, \xi) \overset{\text{def}}{=} \chi(x - \mathbf{n})\chi(\xi - \mathbf{n}')e^{i(\xi - \mathbf{n}) \cdot (x - \mathbf{n})} a(x, \xi) \). We thus get a trace class operator provided

\[
\sum_{n,n'} \| \mathcal{F}_1 K_{n,n'} \|_{L^1} = \sum_{n,n'} \| \mathcal{F}_1 a_{n,n'} \|_{L^1} < \infty
\]

Now, the advantage is that the derivatives of \( a_{n,n'} \) are bounded in terms of those of the derivatives of \( a(x, \xi) \) near \((\mathbf{n}, \mathbf{n}')\), so the above LHS is bounded above by

\[
C \sum_{n,n'} |a|_{2d+1} \| \partial^\alpha a_{n,n'} \|_{L^1} \leq C' \sum_{|\alpha| \leq 2d+1} \| \partial^\alpha a \|_{L^1}.
\]

Restoring without difficulty the factors \( h \), we obtain the following criterion for trace-class property:

**Theorem 3.70.** Assume that the symbol \( a \in S(1) \) satisfies \( \sum_{|\alpha| \leq 2d+1} \| \partial^\alpha a \|_{L^1} < 0 \). Then the operator \( \text{Op}_h^R(a) \) is trace-class, with the bound

\[
\| \text{Op}_h^R(a) \|_{tr} \leq C_d h^{-d} \sum_{|\alpha| \leq 2d+1} \| \partial^\alpha a \|_{L^1(\mathbb{R}^{2d})}.
\]

Its trace is explicitly given by

\[
\text{tr} \text{Op}_h^R(a) = \frac{1}{(2\pi h)^d} \int a(x, \xi) \, dx \, d\xi.
\]

**Corollary 3.71.** The same results hold for any quantization \( \text{Op}_t(a) \), in particular for the Weyl quantization.

**Proof.** Exercise: Check that the \( L^1 \) estimates on \( \partial^\alpha a_1 \) imply the same type of estimates on \( \partial^\alpha a_t = e^{i\hbar(t-1)D_x - \hbar t} \partial^\alpha a_1 \) for any \( t \in [0, 1] \). \( \square \)

4. Qualitative and Quantitative Study of the Spectrum of PDOs

We now want to use the \( h \)-PDO toolbox to get informations on the spectral properties of \( \text{Op}_h(a) \) in terms of the properties of its symbol \( a(x, \xi) \). We’ll see that in some cases we have access to a rather precise description of that spectrum.

4.1. Invertibility of elliptic PDOs and Gårding inequalities. The first question is that of the invertibility of the operator \( \text{Op}_h(a) \), when the symbol \( a \) is invertible (that is, nonvanishing on \( \mathbb{R}^{2d} \)).

**Definition 4.1.** A symbol \( a \in S(1) \) is said to be *(semiclassically)* elliptic if \( |a(\rho)| \geq \gamma > 0 \) for all \( \rho \in \mathbb{R}^{2d} \).

In that case, the pseudodifferential calculus of Cor. 3.58 will allow us to construct a parametrix for \( \text{Op}_h^W(a) \) (that is, a quasi-inverse), which is then easily transformed into a true inverse.
Theorem 4.2. Assume that \( a \in S(1) \) is elliptic. Then for \( h \) small enough, \( \text{Op}_h(a) \) is invertible, and its inverse is a PDO with symbol \( b \in S(1) \) admitting an asymptotic expansion

\[
b \sim \sum_j h^j b_j, \quad \text{with principal symbol } b_0 = \frac{1}{a}.
\]

Proof. We start by the trial function \( b_0 = 1/a \in S(1) \) (due to ellipticity of \( a \)). Then, the calculus shows that

\[
a\#b_0 = 1 + r_1, \quad r_1 = \mathcal{O}(h)_{S(1)}.
\]

So for \( h \) small enough, \( \|\text{Op}_h(r_1)\| < 1/2 \), so we may invert \( 1 + \text{Op}_h(r_1) \) by Neumann series\(^1\)\(^\text{7} \), to get

\[
\text{Op}_h(a) \text{Op}_h(b_0) (I + \text{Op}_h(r_1))^{-1} = I,
\]

which produces a right inverse \( B^R \) for \( \text{Op}_h(a) \), with operator norm \( \|B^R\| \leq C \). On may similarly construct a left inverse of the form \( B^L = (I + \text{Op}_h(r_2))^{-1} \text{Op}_h(b_0) \). The existence of these two inverses shows that \( \text{Op}_h(a) \) is invertible, and that \( B^R = B^L \text{Op}_h(a)B^R = B^R = B \).

To prove that \( B = \text{Op}_h(b) \) with \( b \in S(1) \), one may use Beals’s Theorem 3.60, and the following algebraic trick. For any linear symbols \( \ell_1, \ldots, \ell_N \), the commutator

\[
ad_{\text{Op}_h(\ell_1)} B = -B (ad_{\text{Op}_h(\ell_1)} \text{Op}_h(a)) B = \mathcal{O}(h)_{L^2 \rightarrow L^2}.
\]

Applying the Leibniz rule to this expression, one shows by iteration that

\[
\prod_{j=1}^N ad_{\text{Op}_h(\ell_j)} B = \mathcal{O}(h^N)_{L^2 \rightarrow L^2}.
\]

Beals’s Lemma then states that \( b \in S(1) \).

To get more information on the symbol \( b \), we notice that \( r_1 \in hS(1) \), so we may partially cancel it by taking \( b_1 = -r_1/a \), so that \( a\#b_1 = -r_1 + \mathcal{O}(h^2)_{S(1)} \). Continuing this way, we construct \( b_2, b_3, \ldots, b_N \) with \( b_j \in h^jS(1) \), such that \( a\#(b_0 + \cdots + b_N) = I + \mathcal{O}(h^{N+1})_{S(1)} \). An equivalent way to obtain the expansion of \( b \) is to write:

\[
b \sim b_0\#(1 - r_1 + r_1\#r_1 - r_1\#r_1\#r_1 + \cdots)
\]

\( \square \)

Corollary 4.3. This inversion property of elliptic operators can be generalized to elliptic symbols in classes \( S(m) \) with \( m(\rho) \rightarrow \infty \) as \( |\rho| \rightarrow \infty \).

Proof. We want to invert the operator \( \text{Op}_h(a) \), \( a \in S(m) \) elliptic. From the symbol \( b_0 = a^{-1} \in S(m) \), we have \( a\#b_0 = 1 + r_1(h) \), with the symbol calculus showing that \( r_1(h) \in hS(1) \). We can then use the preceding theorem to invert \( \text{Op}_h(1 + r_1) \) into a PDO with symbol \( \sim 1 - r_1 + r_1\#r_1 - \cdots \), and finally apply the Moyal product with \( a^{-1} \in S(m^{-1}) \) to get an inverse \( b \in S(m^{-1}) \). \( \square \)

An interesting corollary of the invertibility of PDOs concerns bounds on the spectrum of an operator with a positive symbol.

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\(^{17}\)The Neumann series is the expression \((I + R)^{-1} = \sum_{n \geq 0} (-1)^n R^n\), with norm \( \|(I + R)^{-1}\| \leq (1 - \|R\|)^{-1} \), valid as soon as \( \|R\| < 1 \).
**Proposition 4.4.** (Weak Gårding inequality) Assume that \( a \in S(1) \) satisfies \( a \geq 0 \). Then for each \( \epsilon > 0 \) there exists \( h_\epsilon \) such that, for any \( h < h_\epsilon \), the self-adjoint operator \( \text{Op}_h^W(a) \) satisfies
\[
\text{Op}_h^W(a) \geq -\epsilon I.
\]

**Proof.** For any \( z < 0 \) the function \( a(\rho) - z \) is elliptic, so we may try to invert it as above. We expand the Moyal product
\[
(a - z) \# (a - z)^{-1} = 1 + 0 + r_2(z; \hbar).
\]
The estimates on \( r_2(z; \hbar) \) are of order \( \hbar^2 \). They given in terms of a certain number of derivatives of \((a - z) \) and \((a - z\lambda)^{-1} \). Fixing some \( \epsilon > 0 \), we want these estimates to satisfy some uniformity w.r.t. \( z \in (-\infty, -\epsilon] \). The derivatives (w.r.t. \( \rho \)) of \((a(\rho) - z) \) are obviously independent of \( z \). The derivatives of \((a(\rho) - z)^{-1} \) are schematically of the form:
\[
(4.1) \quad \partial^\alpha (a - z)^{-1} = (a - z)^{-1} \sum_{k=1}^{[\alpha]} \sum_{\beta_1 + \cdots + \beta_k = \alpha} C_\beta \prod_{j=1}^k (a - z)^{-1} \partial^{\beta_j} a,
\]
so the RHS is bounded above by \( \frac{|\partial^\alpha a|}{(a-z)^2} + \frac{|\partial^\alpha a||\partial^\alpha a|}{(a-z)^3} + \cdots + \frac{|\partial^\alpha a|^k}{(a-z)^k+1} \), uniformly for all \( z \leq -\epsilon \). As a result, \( \|\text{Op}_h^W(r_2(z; \hbar))\|_{L^2 \to L^2} \leq C\hbar^2 \) for all \( z \leq -\epsilon \) and all \( \hbar \in (0, 1] \). Hence, there exists \( h_\epsilon > 0 \) such that \( I + \text{Op}_h^W(r_2(z; \hbar)) \) is invertible of uniformly bounded inverse for all \( z \leq -\epsilon \) and \( h \in (0, h_\epsilon) \). As a consequence, for all \( h \in (0, h_\epsilon) \) the operators \( \text{Op}_h^W(a) - z \) admit inverses for all \( z \leq -\epsilon \), of the form
\[
(4.2) \quad (\text{Op}_h^W(a) - z)^{-1} = \text{Op}_h^W((a - z)^{-1}) \circ (I + \text{Op}_h^W(r_2(z; \hbar)))^{-1},
\]
hence \( \text{Spec Op}_h^W(a) \subset (-\epsilon, \infty) \). □

Modulo some much more involved work, the above estimate on \( \text{Spec(Op}_h^W(a)) \) can be sharpened into the following theorem.

**Theorem 4.5.** (Sharp Gårding inequality) Assume that \( a \in S(1) \) satisfies \( a \geq 0 \). Then there exists \( C_0 > 0 \) and \( h_0 > 0 \) such that, for any \( h < h_0 \), the self-adjoint operator \( \text{Op}_h^W(a) \) satisfies
\[
\text{Op}_h^W(a) \geq -C_0h.
\]

**Proof.** We will construct an inverse of \( \text{Op}_h^W(a - z) \) for \( z = -\hbar/\tilde{h} \), where \( \tilde{h} > 0 \) is a small parameter independent of \( h \). For this we will use the PDO calculus with symbols the \( h \)-dependent “exotic” symbol class. As in the weak version, we will use the trial function \((a - z)^{-1} \). In order to apply some PDO calculus, we need to estimate the derivatives of this function. The first bound is \( \| (a - z)^{-1} \|_{L^\infty} \leq |z|^{-1} \).

To estimate the derivatives of this function we will use \( (4.1) \). The simple bounds
\[
(4.3) \quad (a - z)^{-1} \left| \partial^\alpha a(x) \right| \leq C_\beta |z|^{-1}
\]
produces the bound
\[
(4.4) \quad \left| \partial^\alpha (a - z)^{-1} \right| \leq C_\alpha (a - z)^{-1} |z|^{-|\alpha|}.
\]
However, we can take advantage of the positivity of \( a \) to significantly improve this upper bound. Indeed, by using the Taylor expansion to 2d order one can easily show the pointwise bound \( |\partial a(x)| \leq
\((2 \| \partial^2 a \|_{L^\infty} a(x))^{1/2}\) This leads to
\[
|z|^{1/2} |\partial a(x)| \leq C |z|^{1/2} a(x)^{1/2} \leq C (a - z),
\]
so that \((a - z)^{-1} |\partial a(x)| \leq C |z|^{-1/2},\)
a clear improvement of (4.3) for \(|\beta| = 1\). For the terms with \(|\beta| \geq 2\) we keep (4.3). Injecting these two estimates in the expansion (4.1) we see that the “worst term” is the term with all \(|\beta| = 1\), which gives the bound
\[
\left| \partial^\alpha (a - z)^{-1} \right| \leq C_\alpha (a - z)^{-1} |z|^{-|\alpha|/2},
\]
much sharper than (4.4). This shows that if we take \(z = -\frac{\hbar}{\tilde{\hbar}}\) the function \(|z| (a - z)^{-1}\) belongs to the exotic symbol class
\[
\tilde{S}_1/2 = \tilde{S}_{1/2}(\mathbb{R}^{2d}) \text{ def } \left\{ a \in C^\infty(\mathbb{R}^{2d}), \forall \alpha \in \mathbb{N}^{2d}, |\partial^\alpha a| \leq C_\alpha \left( \hbar^{-1/2} \tilde{\hbar} \right)^{-|\alpha|/2} \right\}.
\]
Such symbols become quite singular in the semiclassical limit, so it is not clear how to manipulate them and the corresponding operators. Like in the proof of the Calderon-Vaillancourt theorem, a trick consists in applying an appropriate rescaling. Notice that the rescaled function
\[
\tilde{a}(\rho) \text{ def } a \left( \left( \hbar^{-1/2} \tilde{\hbar} \right)^{1/2} \rho \right)
\]
belongs to the class \(S(1)\). A simple computation shows that \(\text{Op}_h^W(a)\) is unitarily equivalent to \(\text{Op}_h^W(\tilde{a})\); applying the C-V theorem 3.56 to \(\text{Op}_h^W(\tilde{a})\) shows that this operator is uniformly bounded on \(L^2 \to L^2\).
Examining the Moyal product expansion in Thm 3.49, we see that if \(a_1, a_2 \in \tilde{S}_{1/2}\), then the term of order \(j\) in the expansion of \(a_1 \#_h a_2\) has a size of order \(\hbar^j \left( \hbar^{-1/2} \tilde{\hbar} \right)^{-j/2} = \tilde{\hbar}^j\), a fact which is quite natural when noticing that this product is conjugate (by rescaling) to \(\tilde{a}_1 \#_h \tilde{a}_2\), with \(\tilde{a}_1 \in S(1)\), this product having an asymptotic expansion in powers of \(\hbar\).

On the other hand, if \(a_1 \in \tilde{S}_{1/2}\) and \(a_2 \in S(1)\), then in the expansion of \(a_1 \#_h a_2\) the \(j\)-th term has a size \(\hbar^j \left( \hbar^{-1/2} \tilde{\hbar} \right)^{-j/2} = \left( \tilde{\hbar} \right)^{-j/2}\), and the remainder is of order \(\left( \tilde{\hbar} \right)^{N/2}\).

Equipped with the exotic calculus, we get
\[
(a - z) \#_h (a - z)^{-1} = 1 + 0 + r_2(z; \hbar), \quad r_2 \in \tilde{\hbar}^2 \tilde{S}_{1/2},
\]
As a result, for \(\hbar \leq h_0, \tilde{\hbar} \leq \tilde{h}_0\), the operator \(I + \text{Op}_h(r_2(z; \hbar))\) is invertible with uniformly bounded inverse. The same holds for product \((a - z)^{-1} \#_h (a - z)\), which shows that \(\text{Op}_h^W (a - z)\) is invertible. Like in the proof of Prop. 4.4, one can show that the bound on \(\text{Op}_h(r_2(z; \hbar))\) is uniform w.r.t. \(\lambda\) in the interval \(z \in (-\infty, -\hbar/\tilde{\hbar}]\), which shows that \(\text{Spec Op}_h^W (a) \subset (-\hbar/\tilde{\hbar}, \infty)\). Calling \(C_0 = 1/\tilde{h}_0\), we have thus shown that \(\text{Op}_h^W (a) \geq -C_0 \hbar\) for \(\hbar < h_0\).

Remark 4.6. One can show that the Weyl quantization actually satisfies the sharper Fefferman-Phong inequality
\[
\text{Op}_h^W (a) \geq -C_0 h^2.
\]
The Weyl quantization maps a real symbol into a selfadjoint operator, but not a positive symbol to a positive operator. However, the above result shows that in the semiclassical limit, \(\text{Op}_h^W (a)\) is “almost
positive\(^3\). We see in TD4 that the anti-Wick quantization automatically satisfies a stricter positivity property: the quantization of a positive symbol is automatically a positive operator.

**Corollary 4.7.** *(Improved norm bound)* For \(a \in S(1)\) real-valued, there exists \(C_a > 0\) and \(h_0 > 0\) such that, for any \(h < h_0\), the operator \(\text{Op}_h^W(a)\) satisfies

\[
\inf_\rho a(\rho) - C_a h \leq \text{Op}_h^W(a) \leq \sup_\rho a(\rho) + C_a h.
\]

In particular,

\[
\|\text{Op}_h^W(a)\| \leq \|a\|_{L^\infty} + C_a h.
\]

**Remark 4.8.** We notice that this estimate is sharper than then bound obtained in the Calderon-Vaillancourt Thm 3.56.

**Proof.** The symbol \(a_-(\rho) \overset{\text{def}}{=} a(\rho) - \inf a \geq 0\), so the sharp Gårding inequality implies that \(\text{Op}_h^W(a_-) \geq -Ch\), proving the lower bound on \(\text{Op}_h^W(a)\). The symbol \(a_+(\rho) = \sup a - a(\rho) \geq 0\), so \(\text{Op}_h^W(a_+) \geq -Ch\), proving the upper bound. \(\square\)

### 4.2. Functional calculus on PDOs.

In the previous section we have made use, for the first time, of the *resolvent* of a selfadjoint operator, namely the operator valued function \(\text{Op}_h(a)\), for proving the upper bound.

We will specifically consider functions of selfadjoint operators \(A = \text{Op}_h^W(a)\), with \(a\) a real valued symbol, not necessarily bounded. In this framework, the functions of \(A\) can be defined using the spectral theorem, as explained in the Appendix (see Corollaries 6.2 and 6.5). For a continuous, bounded function \(f : \mathbb{R} \to \mathbb{R}\), the operator \(f(A)\) is then a bounded selfadjoint operator on \(L^2(\mathbb{R}^d)\), with norm \(\|f(A)\|_{L^2 \to L^2} \leq \|f\|_{C^0}\).

More precisely,

\[
\|f(A)\|_{L^2 \to L^2} = \sup_{t \in \text{Spec } A} |f(t)|.
\]

We want to investigate the “nature” of the operator \(f(A)\) when \(A\) is a PDO and \(f : \mathbb{R} \to \mathbb{R}\) is a smooth, compactly supported function. We will show that it is a nice PDO, which will allow us to develop the symbol calculus of \(f(A)\) (that is, compute the asymptotic expansion for its symbol in terms of the symbol \(a\) and the function \(f\)).

We will use a Cauchy formula (sometimes called the Hellfer-Sjöstrand formula) to define the operator \(f(A)\). This formula uses an *almost analytic extension* of the function \(f\).

**Definition 4.9.** Consider \(f \in C_c^\infty(\mathbb{R}; \mathbb{R})\). An *almost analytic extension* of \(f\) is a function \(\tilde{f} \in C_c^\infty(\mathbb{C}, \mathbb{C})\) which coincides with \(f\) on \(\mathbb{R}\), and is “almost analytic” on \(\mathbb{R}\), in the sense that\(^{18}\)

\[
\forall N \geq 0, \forall z \in \mathbb{C}, \quad \left|\partial \tilde{f}(z)\right| \leq C_N |\Im z|^N.
\]

A short way to write this almost analyticity is \(\partial \tilde{f}(z) = \mathcal{O}(|\Im z|^\infty)\).

\(^{18}\)We remind the notation of holomorphic and anti-holomorphic derivatives. For \(z = x + iy\), \(\partial_x = \frac{1}{2} (\partial_x - i\partial_y)\), \(\bar{\partial}_x = \frac{1}{2} (\partial_x + i\partial_y)\).
There are several ways to define such an extension. One way uses the Fourier transform \( \hat{f} \), and two cutoff functions \( \psi, \chi \in C^\infty_c(\mathbb{R}) \) with \( \psi = 1 \) near supp \( f \), \( \chi = 1 \) near 0. The extension is defined as

\[
\tilde{f}(x + iy) \overset{\text{def}}{=} \chi(y)\psi(x) \int e^{i\xi(x+iy)} \chi(y\xi) \hat{f}(\xi) \frac{d\xi}{(2\pi)^{1/2}}.
\]

This formula converges absolutely for any \( y \in \mathbb{R} \). The property \( \tilde{f}|_{\mathbb{R}} = f \) is easy to check. The property \( \partial \tilde{f}(z) = O(|z|^N) \) can be checked by direct computation of \( \partial \tilde{f} \), a couple of integration by parts, using the fact that \( \psi(x) \) vanishes on the support of \( f \).

**Exercise 4.10.** Check that \( \partial \tilde{f}(x + iy) = O(y^N) \) for any \( N > 0 \).

We can now state the Cauchy formula:

**Proposition 4.11.** Let \( f \in C^\infty_c(\mathbb{C}) \) be an almost analytic extension\(^{19} \) of \( f \). Then for any \( t \in \mathbb{R} \),

\[
(4.5) \quad f(t) = -\frac{1}{\pi} \iint \partial \tilde{f}(x + iy) (x + iy - t)^{-1} \, dx \, dy = -\frac{1}{\pi} \iint \partial \tilde{f}(z) (z - t)^{-1} \, d^2z.
\]

Notice that the integrand is smooth when \( y \to 0 \).

As a result, for any selfadjoint operator \( A \), the function \( f(A) \) can be written as

\[
(4.6) \quad f(A) = -\frac{1}{\pi} \iint \partial \tilde{f}(z) (z - A)^{-1} \, d^2z.
\]

**Proof.** To prove the scalar formula we integrate by parts to get \( \frac{1}{\pi} \iint \tilde{f}(z) \partial(z - t)^{-1} \, d^2z \), and we use the distributional formula \( \partial \frac{1}{z} = \pi \delta(z) \) to conclude. To prove the operator expression, we may write \( A \) using the projection valued decomposition

\[
A = \int \lambda dP_\lambda,
\]

where \( P_\lambda \) is the spectral projector of \( A \) on the interval \((-\infty, \lambda]\), so that for \( z \not\in \mathbb{R} \) we have \((z - A)^{-1} = \int (z - \lambda)^{-1} \, dP_\lambda \). We can then define

\[
-\frac{1}{\pi} \iint \partial \tilde{f}(z) (z - A)^{-1} \, d^2z = -\frac{1}{\pi} \iint \partial \tilde{f}(z) \int (z - \lambda)^{-1} \, dP_\lambda d^2z,
\]

noticing that the integrand is well-defined in the limit \( y \to 0 \) due to the estimate

\[
\left\| (z - A)^{-1} \right\| \leq \frac{1}{|3z|}.
\]

We can apply Fubini’s theorem and for ally \( \lambda \in \mathbb{R} \) apply the expression (4.5) to recover \( \int f(\lambda) \, dP_\lambda = f(A) \). \( \square \)

This expression for \( f(A) \) will allow us to extend our results on the resolvent \((z - A)^{-1}\), onto results for \( f(A) \).

### 4.2.1. Refined estimates for the resolvent

Let us consider a real-valued symbol \( a \in S(m) \), with the assumption \( m \geq 1 \), possibly unbounded, and assume that \((a + i)\) is elliptic in \( S(m) \), that is, \(|a(\rho) + i| \geq \gamma m(\rho)\) for all \( \rho \in \mathbb{R}^{2d} \). In that case, the function \((a + i)^{-1} \in S(m^{-1})\), and this is also the case for any function \((a - z)^{-1} \), \( z \not\in \mathbb{C} \). According to Thm 4.2 and the following Claim, for a given value of \( z \) away from the

\(^{19}\)Actually it is sufficient to require that \( \partial \tilde{f}(x + iy) = O(|y|) \).
range of \( a \), in particular for \( \Im z > 0 \) fixed, the symbol \( b(z) \) of the resolvent \( (z - \Op_h W(a))^{-1} \) belongs to the class \( S(m^{-1}) \), and its principal symbol is the function \( (z - a)^{-1} \). However, because we will need to integrate the resolvent over the \( z \in \text{supp} \tilde{f} \) which contains the real axis, we need to understand how the estimates on \( b(z) \) depend on \( z \), in particular when \( \Im z \to 0 \).

**Lemma 4.12.** Take \( a \in S(m) \). The symbol \( b(\rho; z; \hbar) \) of the resolvent \( B(z) = (z - \Op_h W(a))^{-1} \) satisfies the following bounds, uniformly for \( \hbar \in (0, 1] \), \( z \in \{ |\Re z| \leq R, \Im z > 0 \} \) and \( \rho \in \mathbb{R}^{2d} \):

\[
|\partial^\alpha b(\rho; z; \hbar)| \leq C_{\alpha,a} \max \left( 1, \frac{\hbar^{1/2}}{|\Im z|} \right)^{2d+1} |\Im z|^{-1-|\alpha|}.
\]

**Proof.** We first treat the case \( m = 1 \). We use the inverse-CV result of Eq.(3.55) applied to the symbols \( b(z) \) of \( B(z) = (z - \Op_h W(a))^{-1} \), slightly refining the proof of Beals’s Theorem. An easy computation shows that

\[
\text{ad}_{\Op_h(\ell_N)} \cdots \text{ad}_{\Op_h(\ell_1)} B(z) = \mathcal{O}\left( \frac{\hbar^N}{|\Im z|^{N+1}} \right),
\]

which implies \( \Op_h (\partial^\beta b(\rho; z; \hbar) = \mathcal{O}\left( \frac{1}{|\Im z|^{N+1}} \right) \). Applying the “inverse C-V theorem” (3.55) we find that for any \( \alpha \in \mathbb{N}^{2d} \),

\[
||\partial^\alpha b(z)||_{L^\infty} \leq C_{\alpha} \sum_{|\beta| \leq 2d+1} \hbar^{|eta|/2} |\Im z|^{-1-|\beta|-|\alpha|}.
\]

Depending on the ratio \( \frac{\hbar^{1/2}}{|\Im z|} \), this sum is dominated by either the term \( |\beta| = 0 \) or by the terms \( |\beta| = 2d+1 \), which gives the required result.

Now, if \( a \in S(m) \) with \( m(\rho) \to \infty \) and \( (a + i) \) is elliptic in \( S(m) \). As above we call \( B(z) = (z - A)^{-1} \), \( A = \Op_h W(a) \). When computing the commutator with \( \Op_h(\ell) \), then we apply the argument of Beals’s Theorem to the operator \( P(z) = \left( i - \Op_h W(a) \right) (z - \Op_h W(a))^{-1} \). This operator satisfies the \( L^2 \) bound \( \|P(z)\| \leq \frac{C}{|\Im z|} \). When we commute it with \( \Op_h(\ell) \) we get

\[
[\ell^\nu, B(z)] = B(z) [\ell^\nu, A] B(z) = B(z) [\ell^\nu, A] B(i) (i - A) B(z)
\]

Now, we use the fact that \( \|(i - A) B(z)\| = \mathcal{O}\left( \frac{1}{|\Im z|} \right) \), and that \( [\ell^\nu, A] B(i) \) is an operator with symbol in \( hS(1) \), so that \( \|[\ell^\nu, A] B(i)\| = \mathcal{O}(h) \). Using also that \( \|B(z)\| = \mathcal{O}\left( \frac{1}{|\Im z|} \right) \) we obtain \( \|[\ell^\nu, B(z)]\| = \mathcal{O}\left( \frac{h}{|\Im z|^2} \right) \). Proceeding by iterations we prove the estimate (4.9) for this case as well, which leads to the result. \( \square \)

We notice that if \( |\Im z| \) is too small, \( b(\rho; z; \hbar) \) does not belong to a “good” symbol class. As we have seen above, a good calculus is possible for \( |\Im z| \gg \hbar^{1/2} \), corresponding to the class \( \tilde{S}_{1/2} \). Yet, the above estimates can be used to show that the operator \( f(A) \) is a PDO.

**Corollary 4.13.** For \( a \in S(m) \), the operator \( f(A) \) is a PDO with symbol \( c \in S(1) \).

For \( a \in S(m) \) elliptic, \( f(A) \) is a PDO with symbol \( c \in S(m^{-\infty}) \).
Lemma 4.16. Since each term in the expansion of Claim 4.15 belongs to the symbol class \( S(1) \), another way to express this is to notice that the symbols \( \bar{f}(z) b(z; h) \) belong to \( S(1) \), uniformly w.r.t. \( z \).

For any \( A \in \mathbb{R}^d \), we may apply the Cauchy formula (4.6) at the level of symbols, we get the explicit formula:

\[
(4.10) \quad c(\rho; h) = -\frac{1}{\pi} \int \int \bar{f}(z) b(\rho; z; h) \, d^2 z.
\]

The estimates on \( \bar{f}(z) \) and (4.8) show that \( \int \bar{f}(z) \partial^\alpha b(\rho; z; h) = O(|\Im z|\delta), \) uniformly w.r.t. \( \rho \in \mathbb{R}^d \), so that \( c(h) \in S(1) \). Another way to express this is to notice that the symbols \( \bar{f}(z) b(z; h) \) belong to \( S(1) \), uniformly w.r.t. \( z \).

Proof. If we apply the Cauchy formula (4.6) at the level of symbols, we get the explicit formula:

\[
(4.11) \quad \partial^\alpha (z - a) = -\frac{1}{\pi} \int \int \bar{f}(z) b(\rho; z; h) \, d^2 z,
\]

and we have just shown that

\[
(4.12) \quad c = c_{\mathcal{R}} + O(h^\infty)_{S(1)}.
\]

In the region \( \mathcal{R} \), using the expression (4.1) for the derivatives of \( (z - a)^{-1} \), we get the simple bounds

\[
(4.13) \quad \partial \frac{1}{(z - a)} \leq C_{\alpha, a} m(\rho) \frac{1}{|\Im z|^{-|\alpha|}},
\]

valid uniformly for \( \rho \in \mathbb{R}^d \) and \( |z| \leq R_0 \). Hence, if \( z \) is restricted in the region \( \mathcal{R} \), the function \( (z - a)^{-1} \) belongs to the symbol class \( h^{-\delta} S_{\delta}(m^{-1}) \), where we use the following definition of “exotic” symbol class:

**Definition 4.14.** For any \( \delta \in (0, 1/2) \) and any order function \( m(\rho) \), we define the following “exotic” symbol class:

\[
S_{\delta}(m) \equiv \left\{ a \in C^\infty(\mathbb{R}^d), \forall \alpha \in \mathbb{N}^d, \| \partial^\alpha a(\rho) \| \leq C_{\alpha, a} m(\rho) h^{-|\alpha|\delta} \right\}
\]

This symbol class slightly extends the class \( S(m) \), by the fact that the derivatives of the symbol may grow at algebraic rate when \( h \to 0 \).

**Claim 4.15.** The semiclassical calculus, in particular Thm 3.49 and the C-V Theorem 3.56 for the case \( m = 1 \), can be naturally extended to the classes \( S_{\delta}(m) \). However, we have to adapt the statements a bit. Since each term in the expansion of \( a \# b \) is of order \( h^t h^{-\delta} h^{-t\delta} \), the expansions are effectively expansions in powers of \( h^{1-2\delta} \), in particular the remainders in the expansion are of order \( h^{N(1-2\delta)} \).

**Lemma 4.16.** For \( h \) small enough and \( z \in \mathcal{R} \), the symbol \( (z - a)^{-1} \) and \( b(z; h) \) both belong to the class \( h^{-\delta} S_{\delta}(m^{-1}) \), with uniform estimates w.r.t. \( z \).
Proof. The first part is contained in the estimate (4.12). The estimates (4.8) on derivatives of \( b(z) \) show that \( b(z) \in h^{-\delta}S_\delta(1) \). We can easily improve this by adapting the proof of Corollary 4.3 to the setting of the exotic classe \( S_\delta(m) \), namely by considering the expansion
\[
(z - a) \#_h (z - a)^{-1} = 1 + r_2(h, z).
\]
The symbol calculus in \( S(m) \times S(m^{-1}) \) shows that \( r_2 \in h^{2-3\delta}S_\delta(1) \). The Calderon-Vaillancourt for the class \( S_\delta(1) \) implies that for \( h \) small enough \( \text{Op}_h^W (1 + r_2) \) is invertible, of inverse in \( S_\delta(1) \), so if we multiply that symbol on the left by \( (z - a)^{-1} \) we get the symbol \( b(z) \in h^{-\delta}S_\delta(m^{-1}) \).

This proof also directly provides a way to compute the expansion for \( b(z) \): from the identity
\[
\text{Op}_h^W (z - a) \text{Op}_h^W ((z - a)^{-1}) (I + \text{Op}_h^W (r_2))^{-1} = I,
\]
we get
\[
(4.13) \quad b(z) = (z - a) \# (1 - r_2 + r_2 \# r_2 - \cdots).
\]
We want to keep track of the \( z \)-dependence in the expansion for the remainder \( r_2(h, z) \). This expansion takes the following schematic form:
\[
\begin{align*}
  r_2(z) & \sim -\sum_{j \geq 2} \frac{(ih/2)^j}{j!} a \left( \omega(\overline{D}, \overline{D}) \right)^j (z - a)^{-1} \\
 & \quad \sim \sum_{j \geq 2} h^j (z - a)^{-1-j} D^j a \sum_{k=1}^j (z - a)^{j-k} \sum_{j_1 + \cdots + j_k = j} \prod_{i=1}^k (D^{j_i} a),
\end{align*}
\]
where we factorized by \( (z - a)^{-j} \) so as to exhibit in the numerator polynomials \( q_j(z, \rho) \) in the variable \( z \), of degrees \( \leq j - 1 \):
\[
  r_2(z) \sim \sum_{j \geq 2} h^j (z - a)^{-1-j} q_j(z).
\]
Now, the Moyal product \( b_0 \# r_2 \) expands into
\[
\begin{align*}
  \sum_{\ell \geq 0} h^\ell D^\ell (z - a)^{-1} D^\ell r_2 &= \sum_{\ell \geq 0} h^\ell D^\ell (z - a)^{-1} D^\ell \sum_{j=2}^\infty h^j (z - a)^{-1-j} q_j(z) \\
  &= \sum_{\ell \geq 0, j \geq 2} h^{\ell+j} (z - a)^{-1-\ell} Q_\ell(z) (z - a)^{-1-j} Q_{\ell,j}^1(z) \\
  &= \sum_{\ell \geq 0, j \geq 2} h^{\ell+j} (z - a)^{-2-2\ell-j} Q_{\ell+j}^{(1)}(z).
\end{align*}
\]
In this expression we see that at each order \( k = \ell + j \), the denominators have orders \( 2 + 2\ell + j = 2 + 2k - j \leq 2k \), so we may write
\[
  b_0 \# r_2 \sim \sum_{k \geq 2} h^k (z - a)^{-2k} Q_k^{(1)}(z),
\]
and one can check that the degree of the polynomial \( Q_k^{(1)}(z) \) is \( \leq 2k - 4 \).
The next order $b_0#r_2#r_2$ has the form
\[
\sum_{\ell \geq 0} \sum_{k \geq 2, j \geq 2} \hbar^{\ell+k+j}(z-a)^{-1-2(k+\ell)+j} Q_{k,j,\ell}(z),
\]
so at order $\hbar^m$ the maximal power of the denominator is $1 + 2(k+\ell) + j = 1 + 2(m-j) + j = 1 + 2m - j \leq 2m - 1$, with $m \geq 4$:
\[
b_0#r_2#r_2 \sim \sum_{m \geq 4} \hbar^m (z-a)^{-2m+1} Q_{m}^{(2)}(z).
\]
Notice that this term is of order $\hbar^{4(1-2\delta)+\delta}$. By induction, one can show that the higher powers of $b_0#r_2#n$ produce terms of order $\hbar^m$ with a denominator of power $\leq 2m - 1$.

This shows that the asymptotic series (4.13) of $b(z)$ can be written
\[
(4.14) \quad b(z) \sim (z-a)^{-1} + \sum_{k \geq 2} \hbar^k (z-a)^{-2k} Q_k(z),
\]
with $Q_k(z)$ a polynomial of degree $\leq 2k - 1$. For $z \in \mathcal{R}$ we see that each derivative $\partial^\alpha b(z)$ is dominated by the main term $\partial^\alpha b_0(z) = O(h^{-d(1+|\alpha|)})$, which is compatible with the fact that $b(z) \in h^{-\delta} S_\delta(m^{-1})$.

**Lemma 4.17.** For $h$ small enough and $z \in \mathcal{R}$ the symbol $b(z) \in h^{-\delta} S_\delta(m^{-1})$ admits the expansion (4.14), with uniform estimates w.r.t. $z \in \mathcal{R}$.

The symbol $c_{\mathcal{R}}(\rho)$ of the operator $f(A)_{\mathcal{R}}$ can be obtained by injecting the expansion for $b(z)$ in the Cauchy formula (4.10):
\[
c_{\mathcal{R}} \sim -\frac{1}{\pi} \iint_{\mathcal{R}} \partial \tilde{f}(z) \left[(z-a)^{-1} + \sum_{k \geq 2} \hbar^k (z-a)^{-2k} Q_k(z)\right] d^2z.
\]
Since this symbol is a linear combination of symbols $b(z) \in h^{-\delta} S_\delta(m^{-1})$, we get that $c_{\mathcal{R}} \in h^{-\delta} S_\delta(m^{-1})$. We can extend the integral to $z \in \mathbb{C}$, up to a remainder $O(h^\infty)_{S_{\delta}}$ (see (??)). After this extension it is possible to integrate it by parts w.r.t. the variables $x, y$ (or $z, \bar{z}$).

The first term provides the principal symbol
\[
-\frac{1}{\pi} \iint \partial \tilde{f}(z) (z-a(\rho))^{-1} d^2z = f(a(\rho)),
\]
by applying (4.5) with $t = a(\rho)$. The term of order $\hbar^k$, $k \geq 2$, can also be computed by integration by parts:
\[
-\frac{1}{\pi} \iint \partial \tilde{f}(z) Q_k(z) (z-a)^{-2k} d^2z = -\frac{1}{\pi(2k-1)!} \iint \partial \tilde{f}(z) Q_k(z) (-\partial)^{2k-1} [(z-a)^{-1}] d^2z
\]
\[
= \frac{1}{\pi(2k-1)!} \iint \partial^{2k-1} \left[\tilde{f}(z) Q_k(z)\right] \delta \left[(z-a)^{-1}\right] d^2z
\]
\[
= \frac{1}{(2k-1)!} \partial^{2k-1} \left[\tilde{f}(t) Q_k(t)\right] \big|_{t=a(\rho)}.
\]
We obtain the following
Theorem 4.18. The symbol \( c(h) \) for \( f(\text{Op}_h^W(a)) \) admits an expansion in \( S(1) \),

\[
c \sim \sum_{k \geq 0} h^k c_k(\rho), \quad c_0 = f(a), \quad c_1 = 0.
\]

The \( c_k(\rho) \in S(1) \) are supported in the set

\[
\text{supp } f(a) = \left\{ \rho \in \mathbb{R}^d, \ a(\rho) \in \text{supp } f \right\}.
\]

4.2.2. Decay estimates for the symbol of \( f(A) \). If \( m \to \infty \) and \( (a+i) \) is elliptic in \( S(m) \), then the support of \( f(a) \) is necessarily a compact set, showing that all \( c_k \) are compactly supported.

If \( m = 1 \), this support can be unbounded, and we get \( c \in S(1) \). On the other hand, if we assume that \( \text{dist}(a(\rho), \text{supp } f) \geq 1/C \) for \( \rho \) outside of a bounded set, then the function \( f(a) \), as well as all the \( c_k \), are compactly supported.

In these conditions, it is tempting to believe that the symbol \( c \) is fast decaying and \( O(h^{\infty}) \) outside \( \text{supp } f(a) \). This is what we may show:

Proposition 4.19. With the above assumptions on \( a \) and \( f \), the symbol \( c \) of \( f(\text{Op}_h^W(a)) \) belongs to \( S(\mathbb{R}^d) \), with uniform estimates for \( h \in (0, 1] \). Besides, \( c(\rho; h) \) is essentially supported in \( \text{supp } f(a) \), with estimates

\[
\partial^\alpha c(\rho) = O \left( \left( \frac{h}{\text{dist} (\rho, \text{supp } f(a))} \right)^\infty \right), \quad \text{dist} (\rho, \text{supp } f(a)) \geq \epsilon.
\]

Our general assumption is that there exists some bounded neighbourhood \( \Omega \) of \( \text{supp } f(a) \) and a constant \( C > 0 \), such that \( \text{dist}(a(\rho), \text{supp } f) \geq 1/C \) for all \( \rho \notin \Omega \). Without loss of generality, we may assume that

\[
a(\rho) \geq \text{max supp } f + C, \quad \forall \rho \notin \Omega.
\]

By smoothly modifying \( a(\rho) \) inside \( \Omega \), we may construct an auxiliary symbol \( \tilde{a} \in S(m) \), such that

\[
\begin{align*}
\tilde{a}(\rho) &= a(\rho), & & \rho \notin \Omega, \\
\tilde{a}(\rho) &\geq \text{max supp } f + C/2, & & \rho \in \mathbb{R}^d.
\end{align*}
\]

We then use the following resolvent identity, valid for any \( z \notin \mathbb{R} \):

\[
(\zeta - \text{Op}_h^W(a))^{-1} = (\zeta - \text{Op}_h^W(\tilde{a}))^{-1} + (\zeta - \text{Op}_h^W(a))^{-1} \text{Op}_h^W(a - \tilde{a}) (\zeta - \text{Op}_h^W(\tilde{a}))^{-1}.
\]

We can now inject this decomposition into the Cauchy formula (4.6). Let us consider the first term on the RHS. Due to the range of \( \tilde{a} \), we see that \( (\zeta - \text{Op}_h^W(\tilde{a}))^{-1} \) is invertible for \( \zeta \) in a small enough neighbourhood of \( \text{supp } f \), in particular for \( \zeta \in \text{supp } \hat{f} \), with uniform estimates. Besides, \( (\zeta - \text{Op}_h^W(\tilde{a}))^{-1} \) is a bounded operator, depending holomorphically on \( z \) in \( \text{supp } \hat{f} \), so that

\[
\bar{\partial} (\zeta - \text{Op}_h^W(\tilde{a}))^{-1} = 0, \quad \forall \zeta \in \text{supp } \hat{f}.
\]

By integration by parts, this shows that \( f(\text{Op}_h^W(\tilde{a})) = 0 \). This fact could have been deduced from the abstract spectral decomposition of \( f(\text{Op}_h^W(\tilde{a})) \).
We are now interested in integrating the second term over \( z \), and can switch to its symbol representation
\[
c = -\frac{1}{\pi} \int \partial \bar{f}(z) b(z) \# h (a - \tilde{a}) \# h b(z) \, d^2 z.
\]
Since \((a - \tilde{a})\) is compactly supported, we know from the “Quasilocality Lemma” 3.50 that the symbol \( b^{(2)}(z) \overset{\text{def}}{=} (a - \tilde{a}) \# h b(z) \) is in \( \mathcal{S}(\mathbb{R}^d) \) uniformly for \( z \in \text{supp} \bar{f} \), and admits estimates \( \mathcal{O}\left( \left( \frac{h}{\text{dist}(\rho, \Omega)} \right)^\infty \right)_S \) for \( \rho \) outside \( \Omega \). A Moyal product with the symbol \( \partial \bar{f}(z) b(z) \), which is uniformly in \( \mathcal{S}(1) \), gives a symbol in \( \mathcal{S}(\mathbb{R}^d) \) with the same type of estimates. Finally, integrating over \( z \) gives

**Corollary 4.20.** Assume that \( a \in S(m) \) for some order function \( m \to \infty \) as \( |\rho| \to \infty \), and that \( (a + i) \) is elliptic in \( S(m) \). Then, for any function \( f \in \mathcal{C}^\infty_c(\mathbb{R}) \), the operator \( f (\text{Op}_h^W(a)) \) is a PDO with symbol \( c \in \mathcal{S}(\mathbb{R}^d) \), essentially supported in the bounded set \( \text{supp} f(a) \).

As a result, this operator is trace class, and
\[
\text{tr} f (\text{Op}_h^W(a)) = \frac{1}{(2\pi h)^d} \int c(\rho) d\rho \sim \frac{1}{(2\pi h)^d} \sum_{k \geq 0} h^k \int c_k(\rho) d\rho.
\]
In particular, the principal order term is
\[
\frac{1}{(2\pi h)^d} \int f \circ a(\rho) d\rho.
\]
The same conclusion holds if \( a \in S(1) \) with \( \text{dist}(a(\rho), \text{supp} f) \geq 1/C \) for \( \rho \) outside of a bounded set.

This estimate allows to prove the Weyl’s law for operators \( \text{Op}_h^W(a) \) of this type.

We now assume that for some interval \( I \subseteq \mathbb{R} \) (which was previously \( \text{supp} f \) ), there exists a bounded set \( \Omega \subset \mathbb{R}^d \) such that
\[
\text{dist}(a(\rho), I) \geq C, \quad \forall \rho \notin \Omega.
\]

**Theorem 4.21.** (Semiclassical Weyl’s law) Let some compact interval \( [E_0, E_1] \subseteq I \), and let \( N([E_0, E_1]; h) \) be the number of eigenvalues of \( \text{Op}_h^W(a) \) in \( [E_0, E_1] \), counted with multiplicities. Then we have the following estimates as \( h \to 0 \):
\[
\frac{1}{(2\pi h)^d} \left( V_-([E_0, E_1]) + o(1) \right) \leq N([E_0, E_1]; h) \leq \frac{1}{(2\pi h)^d} \left( V_+([E_0, E_1]) + o(1) \right),
\]
where we define the phase space volumes
\[
V_\pm ([E_0, E_1]) = \lim_{\epsilon \to 0} \text{Vol} \left\{ \rho \in \mathbb{R}^d, \quad a(\rho) \in [E_0 + \epsilon, E_1 - \epsilon] \right\}.
\]

**Proof.** For any \( \epsilon > 0 \), one can construct two smooth functions \( f_\pm \in \mathcal{C}_c^\infty(\mathbb{R}) \), such that
\[
\mathbb{I}_{[E_0 + \epsilon, E_1 - \epsilon]} \leq f_- \leq \mathbb{I}_{[E_0, E_1]} \leq f_+ \leq \mathbb{I}_{[E_0 - \epsilon, E_1 + \epsilon]}.
\]
One then has the inequalities
\[
\text{tr} f_-(A) \leq N([E_0, E_1]; h) \leq \text{tr} f_+(A).
\]
We can apply the trace estimate (4.16) on both bounds, leading to
\[
\int f_-(a(\rho)) \, d\rho - \mathcal{O}_{f_-}(h) \leq (2\pi h)^d N([E_0, E_1]; h) \leq \int f_+(a(\rho)) \, d\rho + \mathcal{O}_{f_+}(h).
\]
\[
\text{Vol} a^{-1}([E_0 + \epsilon, E_1 - \epsilon]) - \mathcal{O}_{f_-}(h) \leq (2\pi h)^d N([E_0, E_1]; h) \leq \text{Vol} a^{-1}([E_0 - \epsilon, E_1 + \epsilon]) + \mathcal{O}_{f_+}(h).
\]
We can take a sequence \( \epsilon = \epsilon_0 \searrow 0 \) slowly enough so that the remainders \( \mathcal{O}_{f_{\pm}}(h) \) still decay, and get the result. \( \square \)

Remark 4.22. If \( E_0 \) and \( E_1 \) are regular energies (meaning that \( da(\rho) \) does not vanish on the energy shells \( a^{-1}(E_i) \)), then these volumes are equal to each other, and we have an asymptotics for \( \mathcal{N}([E_0, E_1]; h) \).

A slight improvement of the method would consist in taking \( f_{\pm} \in S_\delta(\mathbb{R}) \), so as to take \( \epsilon = h^\delta \) in the above bounds. To do that, one needs to check that the functional calculus is still well-behaved for such \( h \)-dependent functions. Essentially, the fact that the leading symbol \( f \circ a(\rho) \in S_\delta(\mathbb{R}^d) \) gives a good hint on the fact that \( c \in S_\delta(\mathbb{R}^{2d}) \).
5. Time evolution of observables and quantum-classical correspondence

5.1. Quantum-classical correspondence. In this section we will study, for a given quantum Hamiltonian $P = \text{Op}_h^W(p)$, the adjoint action of the propagator $U(t) = e^{-itP/h}$ on quantum observables $A = \text{Op}_h^W(a)$. This adjoint action, called the Heisenberg action, is “dual” to the action on quantum states.

5.2. Heisenberg evolution of observables. In quantum mechanics we are not necessarily interested in the full description of a quantum state, but rather on some of its “averages” of the form $\langle \psi, A\psi \rangle$, where $A$ is a quantum observable. When the state is evolving under the Schrödinger equation, the quantum average $\langle \psi(t), A\psi(t) \rangle$ can be computed in two ways: either we analyze the states $\psi(t) = e^{-itP/h}\psi(0)$ and then compute its average w.r.t. the observable $A$, or we instead study the evolution of the operator $A$ through the adjoint action of $e^{-itP/h}$, namely the Heisenberg evolution of the operator $A$, given by

$$A(t) \overset{\text{def}}{=} e^{itP/h} \circ A \circ e^{-itP/h}. \quad (5.1)$$

One could study this operator by separately studying the structure of the propagators $U(t)$, $U(-t)$, and then of the product $U(-t)AU(t)$. This is a possible route, but it is quite technical, since it requires to introduce a novel family of semiclassical operators, called Fourier Integral Operators, to which belong the propagators $U(t)$. This class generalizes the class of semiclassical PDOs, and is closed under composition, so in the end one can analyze $A(t)$.

There is fortunately an alternative route, which bypasses the explicit study of the propagators $U(t)$, and allows to directly analyze $A(t)$, staying inside the class of PDOs. This is the strategy we will expose in this section. We will prove Egorov’s theorem, which estabishes that $A(t)$ is a PDO, and will describe its symbol in terms of the symbol $a$ and the Hamiltonian flow $\Phi^t_p$ generated by the Hamiltonian $p(x, \xi)$. This allows to prove an approximate correspondence between the quantum evolution $A(t)$ and the classical evolution of the observable $a$, given by $a(t) = a \circ \Phi^t_p$.

5.3. Evolution through linear flows: exact quantum-classical correspondence. In this section we will restrict ourselves to very particular Hamiltonians, namely we will assume that $p(x, \xi)$ is a quadratic polynomial of $(x, \xi)$.

As a result, the vector field $X_p$ depends on an affine way on the coordinates, and the flow $\Phi^t_p$ is also affine: $\Phi^t_p$ is represented by a symplectic matrix $M_t \in Sp(d, \mathbb{R})$, composed with the translation by a vector $V_t$. It will be easier to treat both dynamics separately, that is first treat linear polynomials $p_1(x, \xi) = -\xi_1 \cdot x + x_1 \cdot \xi$ for some $V_1 = (x_1, \xi_1) \in \mathbb{R}^{2d}$, and then quadratic ones $p_2(x, \xi) = \langle \rho, Q \rho \rangle$, where $Q$ is a $2d \times 2d$ real symmetric matrix.

5.3.1. Evolution through a linear Hamiltonian. The Hamiltonian $p_1(x, \xi)$ generates the following rigid translation: $\Phi^t_{p_1}(\rho) = \rho + tV_1$. The evolution of an observable is just a rigid translation in phase space: $a(t, \rho) = a(\rho + tV_1)$.

At the quantum level, we will choose an arbitrary $s$-quantization. We notice that the propagator $U(t) = \exp \{-it\text{Op}_h(p_1)/h\}$ is nothing else than the quantum phase space translation operator $T_{iV_1}$ (see (3.8)). For this reason, to compute $U(-t)\text{Op}_s(a)U(t)$ it sounds natural to use the representation of the observable
Op_s(a) in terms of the Heisenberg operators, see Eq. (3.21). The composition relations (3.9) between the Heisenberg operators shows that, for any V_0 ∈ ℝ^{2d}, one has
\[
U(-t) \text{Op}_s(e_{V_0}) U(t) = e^{-ith(1/2-s)ξ_o x_0} T_{-tV_1} T_{tv_0} T_{tv_1}
\]
\[
= e^{-ith(1/2-s)ξ_o x_0} \exp \left\{ \frac{i}{2} \left( -tV_1, V_0 \right) \right\} T_{-tV_1} T_{tv_0} T_{tv_1}
\]
\[
= e^{-ith(1/2-s)ξ_o x_0} \exp \left\{ \frac{i}{2} \left( -tV_1, hV_0 \right) + \frac{i}{2} \left( -tV_1 + hV_0, tv_1 \right) \right\} T_{hv_0}
\]
\[
= \exp \{ -itω(V_1, V_0) \} \text{Op}_s(e_{V_0}).
\]

Injecting this identity in the integral (3.20) we easily get
\[
U(-t) \text{Op}_s(a) U(t) = \int \text{Op}_s(e_{V_0}) \exp \{ -itω(V_1, V_0) \} \hat{a}(V_0) \frac{dV_0}{(2π)^d}.
\]

Now, the Fourier representation (3.19) shows that the above operator is the Weyl quantization of the symbol
\[
a_t(\rho) = \int \exp (iω(V_0, ρ)) \exp \{ -itω(V_1, V_0) \} \hat{a}(V_0) \frac{dV_0}{(2π)^d} = \int \exp (iω(V_0, ρ + tv_1)) \hat{a}(V_0) \frac{dV_0}{(2π)^d} = a(ρ + tv_1).
\]

We have thus shown the following covariance property of the s-quantizations:

**Proposition 5.1. (Covariance of quantization with phase space translations)** For any phase space translation T_{v_1}, any observable a ∈ S′(ℝ^{2d}) and any s ∈ [0, 1], one has the covariance property:
\[
(5.2) \quad T_{-V_1} \text{Op}_s(a) T_{V_1} = \text{Op}_s(a(\bullet + V_1)).
\]

Equivalently, for linear Hamiltonian one has an exact quantum-classical correspondence.

5.3.2. **Evolution through a quadratic Hamiltonian.** In this section we will consider quadratic Hamiltonians
\[
p_2(ρ) = \frac{1}{2} (ρ, Qρ) = \frac{1}{2} (\langle x, Q_{11} x \rangle + 2 \langle x, Q_{12} ξ \rangle + \langle ξ, Q_{22} ξ \rangle).
\]

The vector field is given by
\[
\dot{ρ} = \left( \begin{array}{c}
\dot{x} \\
\dot{ξ}
\end{array} \right) = \left( \begin{array}{cc}
Q_{21} & Q_{22} \\
-Q_{11} & -Q_{12}
\end{array} \right) \left( \begin{array}{c}
x \\
ξ
\end{array} \right) = JQρ.
\]

Hence, the flow is given by the matrix ρ(t) = M_tρ(0), with the matrix M_t = exp(tJQ). A simple computation shows that this matrix is symplectic:
\[
T M_t J M_t = \exp( t^T Q^T J ) J \exp(tJQ) = \exp( -tQJ ) J \exp(tJQ) = J \exp( -tQJ ) J \exp(tJQ) = J.
\]

**Remark 5.2.** The quantization of p_2 is, up to a constant shift in the energy, independent of the choice of quantization. Indeed, the only problem of commutation comes from the term \langle x, Q_{12} ξ \rangle, and one has Op_s(\langle x, Q_{12} ξ \rangle) = ⟨x, Q_{12} \hbar D⟩ − ith(1 − s)trQ_{12} = Op^W_s (⟨x, Q_{12} ξ \rangle) + iℏ(s − 1/2)trQ_{12}. Hence, we get
\[
\text{Op}_s(p_2) = \text{Op}^W_s(p_2) + iℏ(s − 1/2)trQ_{12},
\]
so that the adjoint action of $\text{Op}_h(p_2)$ does not depend on the quantization $s$.

On the other hand, we will need to use the Weyl quantization for the symbol $a(\rho)$ (which is generally not a quadratic Hamiltonian) if we want to ensure exact covariance.

**Theorem 5.3.** (Covariance of the Weyl quantization for quadratic Hamiltonians) Assume that $p_2(x, \xi)$ is a quadratic polynomial, so that the flow $\Phi^t_{p_2}$ is given by a symplectic matrix $M_t$. Then, for any $a \in \mathcal{S}'(\mathbb{R}^{2d})$, the Heisenberg evolution of the operator $\text{Op}_h^W(a)$ satisfies the following covariance property:

$$U(-t) \text{Op}_h^W(a) U(t) = \text{Op}_h^W(a \circ M_t).$$

This property actually extends to arbitrary symplectic matrices $M$, quantized by a corresponding metaplectic operator.

We say that the Weyl quantization is covariant with respect to linear symplectomorphisms.

**Remark 5.4.** As opposed to the case of translations, this exact covariance property is specific to the Weyl quantization, and does not hold for $s$-quantizations, nor does it for the anti-Wick quantization.

For a quadratic Hamiltonian $P_2 = \text{Op}_h^W(p_2)$, the quantum propagator $e^{-itP_2/h}$ belongs to the *metaplectic group* $Mp(d, \mathbb{R})$, which can be seen as the quantization of the symplectic group $Sp(d, \mathbb{R})$. We could write

$$e^{-itP_2/h} = \pm U(M_t).$$

As a first step, we claim that the metaplectic group intertwines covariantly with the quantization of linear symbols:

**Lemma 5.5.** Let $\ell(x, \xi)$ be a linear function of $(x, \xi)$. Then, the metaplectic operators $U(t) = U(M_t)$ satisfy the following covariance identity w.r.t. $\text{Op}_h(\ell)$:

$$U(t)^{-1} \text{Op}_h(\ell) U(t) = \text{Op}_h(\ell_t), \quad \ell_t \overset{\text{def}}{=} \ell \circ M_t.$$

**Proof.** To prove this Lemma we use a simple trick: we differentiate w.r.t. the time parameter $s$ the expression $U(s)^{-1} \text{Op}_h(\ell_{t-s}) U(s)$:

$$\frac{d}{ds} U(s)^{-1} \text{Op}_h(\ell_{t-s}) U(s) = U(M_s)^{-1} \left( \frac{i}{\hbar} [P_2, \text{Op}_h(\ell_{t-s})] - \text{Op}_h(\{p_2, \ell_{t-s}\}) \right) U(M_s).$$

Here we used the fact that the classical evolution can be expressed in terms of the Poisson brackets:

$$\frac{d}{dt} \ell_t = \{p_2, \ell_t\}.\quad \text{Because } p_2 \text{ is quadratic and } \ell_{t-s} \text{ is linear in } (x, \xi), \text{ we have the exact correspondence at the level of commutators:}$$

$$\frac{i}{\hbar} [P_2, \text{Op}_h(\ell_{t-s})] = \text{Op}_h(\{p, \ell_{t-s}\}).$$

---

The metaplectic group is not a faithful representation of $Sp(d, \mathbb{R})$, but only a ray representation (actually a representation up to the sign). $Mp$ is a double cover of $Sp$, a generalization of the fact that $SU(2)$ is a double cover of $SO(3)$: the quantization of a rotation by $2\pi$ is given by $-Id$. 
To check this identity we give two representative examples, which essentially exhaust all nontrivial commutators:

\[ [hD_1 hD_2, x_1] = [hD_1, x_1] hD_2 = -i\hbar (hD_2) = -i\hbar \text{Op}_\hbar (\{\xi_1 \xi_2, x_1\}), \]
\[ [hD_1 hD_1, x_1] [hD_1, x_1] hD_1 + hD_1 [hD_1, x_1] = -i\hbar (2hD_1) = -i\hbar \text{Op}_\hbar (\{\xi_1^2, x_1\}). \]

The time derivative (5.5) is therefore vanishing, so that the expressions at \( s = 0 \) and \( s = t \) are equal:

\[ U(t)^{-1} \text{Op}_\hbar (\ell) U(t) = \text{Op}_\hbar (\ell_t). \]

\[ \square \]

We are now equipped to prove the Theorem 5.3:

**Proof.** If we take \( \ell(x, \xi) = \xi_0 \cdot x - x_0 \cdot \xi = \langle J V_0, \rho \rangle \), we find on the RHS of (5.4)

\[ \ell \circ M_t = \langle J V_0, M_t \rho \rangle = \langle \ell M_t, J V_0, \rho \rangle = \langle J M_{t-1} V_0, \rho \rangle, \]

where we used the characteristic property of symplectic matrices (2.13). By quantizing and exponentiating, we get

\[ U(M_t)^{-1} \text{Op}_\hbar^W (e_{V_0}) U(M_t) = \text{Op}_\hbar^W (e_{V_0} \circ M_t) = \text{Op}_\hbar^W (e_{M_t^{-1} V_0}). \]

Injecting this identity in the integral (3.20) gives

\[ U(M_t)^{-1} \text{Op}_\hbar^W (a) U(M_t) = \int T_{h M_t^{-1} V_0} \hat{a}(V_0) \frac{dV_0}{(2\pi)^d} \]

\[ = \int T_{h V_0} \hat{a}(M_t V_0) \frac{dV_0}{(2\pi)^d}, \]

where we used the fact that \( M_t \) preserves the volume form. Finally, we perform the inverse transformations for the classical observable:

\[ \int e_{V_0}(\rho) \hat{a}(M_t V_0) \frac{dV_0}{(2\pi)^d} = \int e_{M_t^{-1} V_0}(\rho) \hat{a}(V_0) \frac{dV_0}{(2\pi)^d} = \int e_{V_0}(M_t \rho) \hat{a}(V_0) \frac{dV_0}{(2\pi)^d} = a \circ M_t(\rho), \]

which shows that the operator in (5.7) is equal to \( \text{Op}_\hbar^W (a \circ M_t) \).

\[ \square \]

Along the proof we have showed the following group-theoretic property.

**Corollary 5.6.** The metaplectic group is intertwined with the Heisenberg group. For any metaplectic operator \( U(M) \), where \( M \in \text{Sp}(d, \mathbb{R}) \), and any translation vector \( V_0 = (x_0, \xi_0) \), one has

\[ U(M_t)^{-1} T_{V_0} U(M_t) = T_{M_t^{-1} V_0}. \]

The expression (5.3) is an exact form of **quantum-classical correspondence.** If we replace the Weyl quantization by another quantization \( \text{Op}_\hbar \), it only holds *approximately*, namely up to a remainder term of order \( \mathcal{O}_\rho(\hbar) \). This is also the case if one considers nonlinear flows, and is the content of the **Egorov Theorem.**
5.4. Egorov’s theorem: approximate quantum-classical correspondence. We now want to generalize the quantum-classical correspondence to more general dynamics. Let us start by considering compactly-supported Hamiltonians $p(x, \xi) \in C_c^\infty(\mathbb{R}^{2d}, \mathbb{R})$, which we assume independent of $\hbar$. The Hamiltonian vector field $X_p$ is supported on $\text{supp} \, p$, which shows that for any $t \in \mathbb{R}$, the flow $\Phi^t_p : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is equal to the identity outside $\text{supp} \, p$. As a result, for any $t \in \mathbb{R}$ the derivatives $\partial^\alpha \Phi^t_p$ are uniformly bounded on $\mathbb{R}^{2d}$. Yet, simple ODE considerations show that these derivatives may grow exponentially with time:

**Lemma 5.7.** There exists $\lambda > 0$ such that, for any $\alpha \in \mathbb{N}^{2d}$, there exists $C_\alpha > 0$ such that

$$\forall t \in \mathbb{R}, \forall \rho \in \mathbb{R}^{2d}, \quad |\partial^\alpha \Phi^t_p(\rho)| \leq C_\alpha e^{\lambda |\alpha| t}.$$  

Here $\lambda$ must be larger than the maximal Lyapunov exponent for the flow $\Phi^t_p$ (or the flow restricted to the complement of $\text{supp} \, p$).

The quantum Hamiltonian is $P = P_\hbar = \text{Op}_\hbar^W(p)$, a PDO bounded on $L^2$, and the Schrödinger propagator $U(t) = e^{-itP/\hbar}$. We will take a symbol $a \in S(m)$, quantized into $A = A(0) = \text{Op}_\hbar^W(a)$. The operator we want to analyze is $A(t) \overset{\text{def}}{=} U(-t)AU(t)$.

Our main result in this section is Egorov’s theorem

**Theorem 5.8.** (Egorov estimate for compactly-supported Hamiltonian)

Take a Hamiltonian $p \in S(1)$, Weyl-quantized into $P = \text{Op}_\hbar^W(p)$. Take a symbol $a \in S(1)$, and quantize it into $A = \text{Op}_\hbar^W(a)$. Then, for any $T \in \mathbb{R}$ (independent of $\hbar$), and for any $t \in \mathbb{R}$, $|t| \leq T$, the operator $A(t) \overset{\text{def}}{=} e^{itP/\hbar} A e^{-itP/\hbar}$ is a PDO with symbol $a(t; \hbar) \in S(1)$ depending smoothly on $t$, with uniform estimates in the interval $t \in [-T, T]$.

The symbol admits the asymptotic expansion $a(t; \hbar) \sim \sum_j \hbar^j a_j(t)$, with principal symbol $a_0(t) = a \circ \Phi^t_p$, and vanishing subprincipal symbol. The higher-order symbols can be computed explicitly, they depend on derivatives of $a$ composed with $\Phi^s_p$.

**Proof.** It is tempting to take the time derivative of $A(t)$:

$$\frac{d}{dt} A(t) = \frac{i}{\hbar} [P, A(t)] = \frac{i}{\hbar} U(-t) [P, A] U(t),$$

but this does not directly prove that this operator is a PDO. We will use a more clever strategy, which is to approximate $A(t)$ by the quantization of the classical evolution, namely

$$A_0(t) \overset{\text{def}}{=} \text{Op}_\hbar^W(a_0(t)) = \text{Op}_\hbar^W(a \circ \Phi^t_p).$$

Using the chain rule and the bounds (5.8), one easily shows that for any $t \in [-T, T]$ the symbol $a_0(t) \in S(m)$. Its time derivative gives

$$A_0(t) = \frac{d}{dt} A_0(t) = \text{Op}_\hbar^W \left( \frac{d}{dt} A_0(t) \right)$$

$$= \text{Op}_\hbar^W \left( \{p, a_0(t) \} \right)$$

$$= \frac{i}{\hbar} [P, A_0(t)] - \hbar^2 \text{Op}_\hbar^W (r_2(t; \hbar)), \quad \text{where } r_2(t; \hbar) \in S,$$

where we used the expansion in the composition Thm 3.49. The estimates on $r_2(t)$ are uniform for $|t| \leq T$. 


At time $t = 0$ one has $A(0) = A_0(0)$. Our objective is to show that $A_0(t)$ is close to $A(t)$ (and that their difference is in $h^2 S(m)$). For this we will use Duhamel’s trick, which consists in interpolating between $A(t)$ and $A_0(t)$ with the following operator:

$$A(t; s) \overset{\text{def}}{=} U(-s)A_0(t-s)U(s).$$

We see that $A(t; 0) = A(t)$, while $A(t; t) = A_0(t)$. Now we fix the time $t$, but differentiate w.r.t. the parameter $s$:

$$\frac{d}{ds}A(t; s) = \frac{d}{ds}U(-s) \left( \frac{i}{\hbar} [P, A_0(t-s)] - \dot{A}_0(t-s) \right) U(s)
= U(-s) \left( \frac{i}{\hbar} [P, A_0(t-s)] - \dot{A}_0(t-s) \right) U(s)
= \hbar^2 U(-s)R_2(t-s)U(s).$$

From $r_2(t) \in \mathcal{S}$, we get $\|R_2(t)\|_{L^2 \to L^2} \leq C_T$ uniformly for $t \in [-T,T]$. Integrating the above time derivative we get the explicit expression

$$A(t : t) - A(t : 0) = A(t) - A_0(t) = \hbar^2 \int_0^t U(-s)R_2(t-s)U(s) \, ds.$$

Using the triangular inequality (and the fact that $U(t)$ is unitary), we obtain

$$A(t) = A_0(t) + O_T \left( th^2 \right)_{L^2 \to L^2}, \quad |t| \leq T.$$

This result already constitutes a quantum-classical correspondence, in the sense of $L^2 \to L^2$ estimates.

The next steps consist in “correcting” the operator $A_0(t)$, such as to cancel, order by order, the remainders. That is, we replace in (5.9) the interpolation operator by the Ansatz

$$B_1(t) = A_0(t) + \hbar^2 A_2(t), \quad A_2(t) = O\mathcal{P}_n^W(a_2(t)), \quad a_2(0) = 0.$$

Injecting this Ansatz in the time derivative, we obtain the extra term $\hbar^2 U(-s) \left( \frac{i}{\hbar} [P, A_2(t-s)] - \dot{A}_2(t-s) \right) (t-s)U(s)$. We want this term to partially cancel the remainder $\hbar^2 U(-t+s)R_2(t-s)U(t-s)$, so we need to minimize the sum

$$\frac{i}{\hbar} [P, A_2(t-s)] - \dot{A}_2(t-s) + R_2(t-s).$$

As a result, we will solve the following inhomogeneous transport equation for the symbol $a_2$:

$$\dot{a}_2(s) - \{p, a_2(s)\} = r_2(s), \quad a_2(0) = 0.$$

This equation is a transport equation, and can be solved by integrating over the trajectories of $\Phi_p^t$. If we call $\tilde{a}_2(s) = a_2(s) \circ \Phi_p^{-s}$, we get $\frac{d}{ds}\tilde{a}_2(s) = \tilde{a}_2(s) \circ \Phi_p^{-s} - \{p, a_2(s)\} \circ \Phi_p^{-s}$, so the above equation amounts to

$$\frac{d}{dt} \tilde{a}_2(s) = r_2(s) \circ \Phi_p^{-s}.$$

Taking into account that the initial condition, we obtain the explicit solution

$$a_2(t) = \int_0^t ds \, r_2(s) \circ \Phi_p^{t-s}.$$
This symbol is in $a_2(t) \in S(1)$, uniformly for $t \in [-T,T]$, and it is essentially supported in $\text{supp}\, p$. We then define $A_2(t) = \text{Op}_h^W(a_2(t))$. The symbol calculus shows that

$$A_2(t-s) - \frac{i}{\hbar} [P, A_2(t-s)] = R_2(t-s) - \hbar^2 R_4(t-s), \quad \text{for some } R_4 = \text{Op}_h^W(r_4), \quad r_4 \in S(1).$$

As a result, we have easily

$$A(t) - A_0(t) - \hbar^2 A_2(t) = \hbar^4 \int_0^t U(-s) R_4(t-s) U(s) \, ds = O(\hbar^4)_{L^2 \to L^2}.$$ 

By iteration, we may construct symbols $a_{2k} \in S$ such that

$$A(t) = \sum_{k=0}^{N-1} \hbar^{2k} A_{2k}(t) + \hbar^{2N} \int_0^t U(-s) R_{2N}(t-s) U(s) \, ds, \quad R_{2N}(s) = \text{Op}_h^W(r_{2N}(s)), \quad r_{2N} \in S(1).$$

To show that $A(t)$ is a PDO, we will apply Beals’s Theorem 3.60. For this we need to estimate the norms of multiple commutators $\text{ad}_{\text{Op}_h^W(a_1)} \cdots \text{ad}_{\text{Op}_h^W(a_N)} A(t)$. We know that each $A_{2k}(t)$ is a PDO, so the norms $\|\text{ad}_{\text{Op}_h^W(a_1)} \cdots \text{ad}_{\text{Op}_h^W(a_N)} \hbar^{2k} A_{2k}(t)\| = O(\hbar^{2k+n})$, which is fine.

There remains to compute the commutators with the integral remainder

$$R_{2N} \overset{\text{def}}{=} \hbar^{2N} \int_0^t U(-s) R_{2N}(t-s) U(s) \, ds, \quad \text{with } r_{2N}(t-s) \in S(1).$$

Let us study the first commutator

$$\text{ad}_\ell (U(-s) R_{2N}(t-s) U(s)) = \text{ad}_\ell (U(-s)) R_{2N}(t-s) U(s) + U(-s) \text{ad}_\ell (R_{2N}(t-s) U(s)) + U(-s) R_{2N}(t-s) \text{ad}_\ell U(s).$$

The Moyal product on $S(1) \times S(1)$ shows that the middle term in the RHS is of order $O(\hbar)$. To treat the two other terms, we need to estimate the norm $\text{ad}_\ell U(s)$. We use a little algebraic trick, namely we compute the time derivative of the operator $(\text{ad}_\ell U(t)) U(-t)$:

$$\frac{d}{dt} \{ (\text{ad}_\ell U(t)) U(-t) \} = \text{ad}_\ell \left( \frac{i}{\hbar} U(t) P \right) U(-t) + \frac{i}{\hbar} \text{ad}_\ell U(t) \, PU(-t)$$

$$= -\frac{i}{\hbar} \left( \text{ad}_\ell (U(t)) P + U(t) \text{ad}_\ell P - \text{ad}_\ell (U(t)) \, P \right) U(-t)$$

$$= -\frac{i}{\hbar} \left( U(t) \text{ad}_\ell PU(-t) \right).$$

Since $\text{ad}_\ell P = O(\hbar)$, this expression is of order $O(1)_{L^2 \to L^2}$, and by integrating over time we find that $\text{ad}_\ell U(t) = O(1)_{L^2 \to L^2}$, uniformly for $|t| \leq T$. We thus obtain

$$\text{ad}_\ell R_{2N} = O(\hbar^{2N}).$$

By iteration, one can show that this estimate does not improve (or worsen) if we iterate the commutators:

$$\text{ad}_{\text{Op}_h^W(a_1)} \cdots \text{ad}_{\text{Op}_h^W(a_N)} R_{2N} = O(\hbar^{2N}).$$

Hence, equipped with the expansion (??) up to order $\hbar^{2N}$, we can check the criterion in Beals’s theorem up to $n = 2N$ commutators. Since the order $2N$ of our expansion can be taken arbitrary large, this criterion holds for any $n \in \mathbb{N}$, which proves that the symbol of $A(t)$ is in $S(1)$. \qed
6. Appendix: Reminder on operator and spectral theory (on Hilbert space)

Below we describe a few properties of spectral theory on $\mathcal{H}$ a separable Hilbert space (we’ll be mostly interested in the case $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^d)$).

6.1. Reminder: spectral theory of bounded operators. Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded operator. Its resolvent set
\[
\rho(A) = \{ z \in \mathbb{C} : (A - z) \text{ is invertible on } \mathcal{H}, \text{ with bounded inverse} \}.
\]
Its spectrum $\text{Spec}(A) = \mathbb{C} \setminus \rho(A)$. The spectrum can be composed of pure point spectrum (eigenvalues) and essential spectrum.

A bounded operator $A$ admits an adjoint $A^*$, which is also bounded. $A$ is selfadjoint iff $A = A^*$.

**Theorem 6.1.** (Spectral theorem) For $A$ a bounded selfadjoint operator, there exists a probability space $(X, \mathcal{M}, \mu)$, a unitary operator $U : \mathcal{H} \to \mathcal{L}^2(X, \mu)$ and a function $f \in \mathcal{L}^\infty(X, \mu)$ such that
\[
A = U^* M_f U,
\]
where $M_f$ is the multiplication by $f$.

Note that the pure point spectrum corresponds to the countable set $\{f(x_i), x_i \text{ an atom of } \mu\}$.

**Corollary 6.2.** (Functional calculus of bounded selfadjoint operators) Take $\theta : \mathbb{R} \to \mathbb{R}$ a continuous function. Then, for $A$ a bounded selfadjoint operator on $\mathcal{H}$, if we represent $A$ as in (6.1), then the function $\theta \circ f \in \mathcal{L}^\infty(\mu)$. We may then define the operator $\theta(A)$ as follows:
\[
\theta(A) = U^* M_{\theta f} U.
\]

One can easily check that this definition is compatible with more obvious one, in the case where $\theta$ is a polynomial.

We remind that an operator $A : \mathcal{H} \to \mathcal{H}$ is compact iff it maps the unit ball $\{\|u\| \leq 1\}$ into a precompact set of $\mathcal{H}$ (that is, a set with compact closure). Compact operators are rather similar with operators of finite rank. In particular, their nonzero spectrum is exclusively made of eigenvalues of finite multiplicities, which may accumulate only at zero.

6.2. Reminder: unbounded selfadjoint operators. An unbounded operator $P$ is defined by its domain $\mathcal{D}(P) \subset \mathcal{H}$, which is assumed to be dense in $\mathcal{H}$. This operator is closable if there exists a closed operator $\bar{P}$ which contains $P$ (meaning that $\mathcal{D}(P) \subset \mathcal{D}(\bar{P})$ and they coincide on $\mathcal{D}(P)$).

The adjoint $P^*$ (and its domain $\mathcal{D}(P^*)$) is defined by duality: $v \in \mathcal{D}(P^*)$ if
\[
|\langle v, Pu \rangle| \leq C(v)\|u\|, \quad \text{for all } u \in \mathcal{D}(P).
\]
Then, one may define $P^* v$ by duality and density of $\mathcal{D}(P)$: due to the above inequality, there exists a unique state $P^* v \in \mathcal{H}$ such that $\langle v, Pu \rangle = \langle P^* v, u \rangle$ for all $u \in \mathcal{D}(P)$. This operator is always closed.

If $P^*$ is densely defined, then $P$ is closable, and its closure can be obtained as $\bar{P} = (P^*)^*$.

$P$ is symmetric if $P \subset P^*$: for all $u, v \in \mathcal{D}(P)$, $\langle v, Pu \rangle = \langle Pv, u \rangle$.

$P$ is essentially selfadjoint if $\bar{P} = P^*$. 
$P$ is selfadjoint if $P = P^*$.

**Theorem 6.3.** (Spectral theorem for unbounded selfadjoint operators) Let $(A, \text{Dom}(A))$ be an unbounded selfadjoint operator on $\mathcal{H}$, with dense domain. There exists a measure space $(X, \mathcal{M}, \mu)$, a unitary operator $U: \mathcal{H} \to L^2(X, \mu)$ and a real valued measurable function $f$ such that

- $\psi \in \text{Dom}(A)$ iff $U\psi \in \text{Dom}(Mf)$, meaning that $MfU\psi \in L^2(X, \mu)$
- if this is the case, then $A\psi = U^*MfU\psi$.

(6.2) 
$A = U^*MfU$.

The domain $\text{Dom}(A)$ corresponds through $U$ to the domain of the multiplication operator $Mf$ on $L^2(X, \mu)$.

The easiest example is that where $A$ is already a multiplication operator. For instance, the operator of multiplication by $x_1$, acting on $L^2(\mathbb{R}^d)$, admits as domain the subspace $\{u \in L^2(\mathbb{R}^d), x_1u \in L^2(\mathbb{R}^d)\}$. We can take $X = \mathbb{R}^d$

**Example 6.4.** Consider the Schrödinger operator of a free particle on $\mathbb{R}^d$, $P_\hbar = -\hbar^2\Delta$. This operator is unbounded, its domain is the Sobolev space $H^2(\mathbb{R}^d)$. Through the (unitary) Fourier transform $\mathcal{F}_\hbar$ it is mapped into the multiplication operator by $|\xi|^2$ acting on the space $L^2(\mathbb{R}^d, d\xi)$. So we may write

$$-\hbar^2\Delta = \mathcal{F}_\hbar^*M_{|\xi|^2}\mathcal{F}_\hbar.$$

**Corollary 6.5.** (Functional calculus of unbounded selfadjoint operators) Take $\theta: \mathbb{R} \to \mathbb{R}$ a continuous bounded function. Then, for $A$ an unbounded selfadjoint operator on $\mathcal{H}$, if we represent $A$ as in (6.2), then the function $\theta \circ f \in L^\infty(\mu)$. We may then define the operator $\theta(A)$ as:

$$\theta(A) = U^*M_{\theta \circ f}U.$$

This operator can be extended to $\mathcal{H}$, where it is bounded, with $\|\theta(A)\|_{L^2 \to L^2} \leq \|\theta\|_{C^0(\mathbb{R})} = \sup_{t \in \mathbb{R}} |\theta(t)|$.

**Theorem 6.6.** (Stone's theorem) Suppose $(A, \text{Dom}(A) \subset \mathcal{H})$ is a selfadjoint (possibly unbounded) operator. Then the function $t \mapsto U(t) = e^{-itA}$ forms a strongly continuous unitary group on $\mathcal{H}$:

$$U(t)U(s) = U(t + s), \quad U(t)^* = U(-t),$$

$$\forall \psi \in \mathcal{H}, \quad ||U(t)\psi - \psi|| \xrightarrow{t \to 0} 0.$$

Furthermore, for any $\psi_0 \in \text{Dom}(A)$, the family of states $\psi(t) = U(t)\psi_0$ solves the Schrödinger equation

$$i\partial_t \psi(t) = A\psi(t), \quad \psi(0) = \psi_0,$$

and one has

$$\frac{U(t)\psi_0 - \psi_0}{t} \xrightarrow{t \to 0} -iA\psi_0.$$

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21In Reed-Simon this calculus is extended to bounded Borel functions, which are functions (i.e. everywhere defined on $\mathbb{R}$) $\theta(t)$ such that for any open interval $I$ the set $\theta^{-1}(I)$ is a Borel set.
References


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