A few aspects of quantum chaotic scattering

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Right: resonant state for the dielectric stadium cavity, computed by C.Schmit.
Outline

• scattering wave (quantum) systems \( \rightsquigarrow \) (complex-valued) resonance spectra, metastable states

• Semiclassical (high-frequency) limit \( \rightarrow \) need to understand the ray dynamics. Importance of the set of trapped classical trajectories.

• A toy model: open quantum maps
  – fractal Weyl law
  – resonance-free strip for filamentary trapped sets
  – phase space distribution of metastable states

• Another class of “leaky” quantum systems: \textit{partially open} systems
  – clustering of decay rates near a \textit{typical} value;
  – fractal Weyl laws
Scattering systems with hard obstacles/smooth localized potential/noneuclidean metric.

- classical dynamics: geodesic (or Hamiltonian) flow + reflection on obstacles. Most rays escape to infinity.

- quantum dynamics: wave or Schrödinger equation governed by $-\Delta_{out}$, resp. $(or \, P_{\hbar} = -\hbar^2 \Delta + V(x))$
For any $E > 0$ the energy shell $\{(x, \xi), \ |\xi|^2 = E\}$ is unbounded, so $-\Delta_{out}$ has a purely continuous spectrum on $\mathbb{R}^+$. 

- $(-\Delta_{out} - k^2)^{-1} : L^2_{comp} \rightarrow L^2_{loc}$ admits a meromorphic continuation from $\{\text{Im } k > 0\}$ to $\{\text{Im } k < 0\}$. Its poles $\{k_j\}$ (of finite multip.) are the resonances of $-\Delta_{out}$.

- Resonances = evals of a nonselfadjoint operator $-\Delta_{out, \theta}$ obtained from $-\Delta_{out}$ by a complex dilation (away from interaction zone)

- Each $k_j$ is associated with a metastable (non-normalizable) state $\psi_j(x)$, with decay rate $\gamma_j = 2|\text{Im } k_j| \leftrightarrow$ lifetime $\tau_j = (2|\text{Im } k_j|)^{-1}$.

$\Rightarrow$ long-living resonance if $\text{Im } k_j = \mathcal{O}(1)$. 

Resonance spectrum
Semiclassical limit

We will focus on the high-frequency limit $\text{Re } k \approx K \gg 1 \Rightarrow$ (micro)localized wavepackets propagate along *classical rays*.

Take $\hbar_{\text{eff}} \overset{\text{def}}{=} K^{-1} \sim$ equivalent to study the resonances $\{z_i(\hbar)\}$ of $\hbar$-dependent operators

$$P_\hbar = -\hbar^2 \Delta_{\text{out}}, \quad \text{more generally} \quad P_\hbar = -\hbar^2 \Delta + V(x)$$

in a disk $D(E, \gamma \hbar)$ centered on a “classical energy” $E$.

High-frequency $\iff$ semiclassical limit $\hbar \ll 1$. 
Main questions we will consider in the semiclassical limit:

- distribution of long-living resonances ($|\text{Im } z_j| = O(\hbar)$)
- phase space localization of metastable modes $\psi_j(\hbar)$
- (time decay of the local intensity $|\psi(x,t)|^2$ (resolvent estimates))

**Main idea:** the distribution of long-living resonances depends on the properties of long classical trajectories. Dispersion of the wave (due to the uncertainty principle) must also be taken into account.

→ relevance of the set of trapped trajectories:

$$\Gamma^\pm = \{(q,p) : \phi^t(q,p) \not\to \infty, \ t \to \mp \infty\}, \quad \Gamma = \Gamma^+ \cap \Gamma^-$$

Long-living resonances represent *quantum mechanics living on* $\Gamma$. 
• We will focus on systems for which the classical flow on $\Gamma$ is strongly chaotic (uniformly hyperbolic: Axiom A system). Such systems are not Liouville-integrable (no conserved quantity except $E$), but their long-time dynamics is well-understood. The trapped set $\Gamma$ is a hyperbolic repeller with fractal geometry.

Semiclassical approach to quantum chaos: identify the appropriate classical-dynamical tools able to provide information on the quantum system.
The ray dynamics can be analyzed through the **return map** $\kappa$ through a **Poincaré section** $\Sigma$.

This map is defined on a subset $\Sigma' \subset \Sigma$, and preserves the induced symplectic form. It is an **Axiom A homeomorphism** on the trapped set $\Gamma \cap \Sigma$.

**Ex:** the bounce map on the obstacles

\[
(q, p = \sin \phi) \mapsto \begin{cases}
\kappa(q, p) = (q', p') \\
\infty
\end{cases}
\]

**Generalization:** consider an arbitrary **symplectic chaotic diffeomorphism** $\tilde{\kappa}$ on some compact phase space (e.g. the torus $\mathbb{T}^2$), and an arbitrary hole $H$ through which particles escape “to infinity” $\mapsto$ **open map** $\kappa = \tilde{\kappa}|_{\mathbb{T}^2 \setminus H}$. 
A toy model: open quantum maps

How to “quantize” such a map $\kappa$? First, define quantum mechanics on $\mathbb{T}^2$:

- Hilbert space $\mathcal{H}_\hbar \equiv \mathbb{C}^N$, $N \sim \hbar^{-1}$

- quantization of observables: $f(q,p) \mapsto \text{Op}_\hbar(f)$ (Pseudodifferential Operator)

- quantization of the diffeom $\tilde{\kappa}$ (various recipes): $U = U_\hbar(\tilde{\kappa})$ unitary matrix (Fourier Integral Operator).

**Quantum-classical correspondence** (until the Ehrenfest time $T_{Ehr} = \frac{|\log \hbar|}{\Lambda}$):

$$
U^{-t} \text{Op}_\hbar(f) U^t = \text{Op}_\hbar(f \circ \tilde{\kappa}) + \mathcal{O}(\hbar e^{\Lambda t}) \quad \text{[Egorov]}
$$

Equivalently, for a wavepacket $|q,p\rangle$, we have $U|q,p\rangle \approx |\tilde{\kappa}(q,p)\rangle$.

To open the “hole”: apply a “projector” $\Pi = \text{Op}_\hbar(1_{\mathbb{T}^2\setminus H})$.

$$
\implies \text{open quantum map} \quad M_N(\kappa) = M_\hbar(\kappa) \overset{\text{def}}{=} \Pi \circ U_\hbar(\tilde{\kappa}) \quad (N \times N \text{ subunitary})
$$
Correspondence with scattering resonances

The spectrum $\{(\lambda_{i,N}, \psi_{i,N}) \mid i = 1, \ldots, N\}$ of the open map $M_\hbar(\kappa)$ should provide a good model for resonances of $P_\hbar$ (numerically much easier).

We expect the statistical correspondence:

$$\{(\lambda_{i,N}, i = 1, \ldots, N) \leftrightarrow \{e^{-iz_j(\hbar)/\hbar}, |\text{Re} z_j(\hbar) - E| \leq \gamma\hbar\}, N \sim \hbar^{-1}\}$$

In particular, the decay rates $\{-2 \text{Im} z_j(\hbar)/\hbar\} \leftrightarrow \{-2 \log |\lambda_{i,N}|\}$.

- To compute resonances of $P_\hbar$, one can actually construct a family of quantum maps $M_\hbar(z)$ associated with the Poincaré return map, such that $\{z_j(\hbar)\}$ are obtained as the roots of $\det(1 - M_\hbar(z)) = 0$ [N-Sjöstrand-Zworski’09?].
Example of an open chaotic map

Dig a rectangular hole in the 3-baker's map on $\mathbb{T}^2$

Advantage: the trapped sets $\Gamma^{(\pm)}$ are simple Cantor sets (simple symbolic dynamics)

$$M_h(B) = F_N^{-1} \begin{pmatrix} F_{N/3} & 0 \\ 0 & F_{N/3} \end{pmatrix}, \quad F_M = \text{discrete Fourier transform}$$
Fractal Weyl law

The *geometry* of the trapped set influences the semiclassical density of long-living resonances.

**Ex:** 2 convex obstacles ⇒ $\Gamma =$ single unstable periodic orbit.

**Quantum normal form** $\rightsquigarrow$ quasi-lattice of resonances [IKAWA, GÉRARD, SJÖSTRAND, ..]

How about a fractal repeller $\Gamma$?

**Theorem.** [SJÖSTRAND’90, SJÖSTRAND-ZWORSKI’05] *In the semiclassical limit, the density of resonances is bounded from above by a fractal Weyl law*

$$
\# \{ j : |z_j(\hbar) - 1| \leq \gamma \hbar \} = \mathcal{O}(\hbar^{-\nu}), \quad \text{resp.} \quad \# \{ j : |\lambda_{j,N}| \geq c \} = \mathcal{O}(N^\nu)
$$

where $\dim_{\text{Mink}}(\Gamma) = 2\nu + 1$ (resp. $= 2\nu$).

**Main idea:** after a suitable transformation, long-living resonant states “live” in a $\sqrt{\hbar}$-nbhd of $\Gamma \rightsquigarrow$ count the number of $\hbar^d$-boxes in this nbhd.

**Conjecture:** $= \mathcal{O}(\hbar^{-\nu})$ should be replaced by $\sim C\gamma \hbar^{-\nu}$
• Such a fractal Weyl law has been numerically confirmed for various systems. Ex: an asymmetric open baker’s map ($\nu$ known explicitly).

• This law was proven for an alternative solvable quantization of the open baker’s map [N-Zworski’05].

• To understand the factor $C_\gamma$ (shape of the curve), an ensemble of random subunitary matrices $(\Pi U)_{U \in COE}$ was proposed in [Schomerus-Tworzydlo’05]. Universal?
Resonance-free strip for “filamentary” repellers

Another dynamical “tool” associated with the flow on $\Gamma$: the topological pressure

$$\mathcal{P}(s) = \mathcal{P}(-s \log J^+) \overset{\text{def}}{=} \lim_{t \to \infty} \frac{1}{t} \log \sum_{p : T_p \leq t} J^+(p)^{-s}$$

“Compromise” between the complexity of the trapped set (# periodic orbits) and the instability of the flow along those orbits.

Properties: $\mathcal{P}(0) = h_{\text{top}}(\Phi^t_{|\Gamma}) > 0$ and $\mathcal{P}(1) = -\gamma_{\text{cl}} < 0$ the classical decay rate.

Theorem. [Ikawa’88, Gaspard-Rice’89, N-Zworski’07] Assume the topological pressure $\mathcal{P}(1/2) < 0$, and take any $0 < g < -\mathcal{P}(1/2)$.
Then, for $\hbar > 0$ small enough, the resonances $z_j(\hbar)$ close to $E$ satisfy $\text{Im} z_j(\hbar) \leq -g \hbar$.

- In dimension $d = 2$, the dynamical condition $\mathcal{P}(1/2) < 0$ is equivalent with the geometrical condition $\text{dim}(\Gamma) < 2$

A too thin repeller disperses the wave.
Analogous results on hyperbolic manifolds

\( X = G \backslash \mathbb{H}^{n+1} \) convex co-compact (infinite volume). The trapped set \( \Gamma \) of the geodesic flow has dimension \( 2\delta + 1 \), where \( \delta \) is the dim. of the limit set \( \Lambda(G) \), as well as the topological entropy of the flow.

Resonances \( s(n - s) = \frac{n^2}{4} + k^2 \) of \( \Delta_X \) are given by the zeros of \( Z_{\text{Selberg}}(s) \) (quantum resonances ↔ Ruelle resonances)

[\text{Patterson’76, Sullivan’79, Patterson-Perry’01}]: all the zeros are in the half-plane \( \text{Im} \, k \leq \delta - n/2 = \mathcal{P}(1/2) \).

This upper bound can be slightly sharpened, and lower bounds for the gap can be obtained [\text{Naud’06,’08}]
Phase space distribution of metastable states

The metastable states \((\psi_j(\hbar))\) associated with long-living resonances have specific phase space distributions.

Consider a family of metastable (normalized) states \((\psi_{i_N})_{N \to \infty}\) of \(M_N(\kappa)\) s.t. the corresponding resonances \(|\lambda_{i_N}| \geq c > 0\). Up to extracting a subsequence, assume that \((\psi_{i_N})\) is associated with a \textit{semiclassical measure} \(\mu\):

\[
\forall f \in C^\infty(T^2), \quad \langle \psi_{i_N}, \text{Op}_\hbar(f)\psi_{i_N}\rangle \xrightarrow{N \to \infty} \int_{T^2} f \, d\mu.
\]

Then for some \(\lambda \geq 0\) we have

\[
|\lambda_{i_N}| \xrightarrow{N \to \infty} \lambda \quad \text{and} \quad \kappa^* \mu = \lambda^2 \mu.
\]

\(\mu\) is a \textit{conditionally invariant measure} with decay rate \(\lambda^2\).
Phase space distribution of metastable states (2)

Condit. invar. measures are easy to construct. They are supported on $\Gamma^+$.

Questions inspired by quantum ergodicity \cite{N-Rubin05, Keating-Novaes-Prado-Sieber06}:

For a given rate $\lambda^2$, which condit. invar. measures $\mu$ are favored (resp. forbidden) by quantum mechanics?

Very partial results for the solvable quantized open baker \cite{Keating-Novaes-N-Sieber08}:

- unique semiclassical measure at the edges of the nontrivial spectrum
- but not in the “bulk” of the spectrum (large degeneracies)
Partially open wave systems

Let us now consider systems for which rays do not escape, but *get damped*.

- **Left**: damped wave equation inside a closed cavity, \((\partial_t^2 - \Delta_{in} + b(x)\partial_t)\psi(x, t) = 0\), \(b(x) \geq 0\) damping function
  \(\leadsto\) spectrum of *complex eigenvalues* \((\Delta_{in} + k^2 + i b(x) k)\psi(x) = 0\)

- **Right**: dielectric cavity. *Resonances* satisfy \((\Delta + n^2 k^2)\psi = 0\), with appropriate boundary conditions \(\leadsto\) reflection/refraction of incoming rays (Fresnel’s laws).

In both cases, the *intensity* \((\Leftrightarrow \text{energy})\) of the rays is reduced along the flow.
\(\rightarrow\) *Weighted ray dynamics.*
Damped quantum maps

Starting from a diffeom. $\tilde{\kappa}$, one can cook up a damped quantum map:

$$M_\hbar(\tilde{\kappa}, d) \overset{\text{def}}{=} \text{Op}_\hbar(d) \circ U_\hbar(\tilde{\kappa}),$$

where $0 < \min |d| \leq |d(q, p)| \leq \max |d| \leq 1$ is a smooth damping function.

⇒ Bounds on the distribution of decay rates of $M_\hbar(\tilde{\kappa}, d)$:

- obvious: all $N$ eigenvalues satisfy $\min |d| \leq |\lambda_{i,N}| \leq \max |d|$ (all resonances in a strip)

- Egorov $\Rightarrow M_\hbar^n \approx U^n \circ \text{Op}_\hbar((d_n)^n)$, where we used the $n$-averaged weights

$$d_n(q, p) \overset{\text{def}}{=} \left( \prod_{j=1}^{n} d(\tilde{\kappa}^j(q, p)) \right)^{1/n}$$

⇒ all evals contained in the (often thinner) annulus $\min |d_\infty| \leq |\lambda_{i,N}| \leq \max |d_\infty|$. 
Taking the chaos into account: clustering of decay rates

Assume \( \tilde{\kappa} \) Anosov \( \Rightarrow \) sharper bounds on the decay rate distribution.

\textbf{Ergodicity} + Central Limit Theorem for \( d_n \Rightarrow \) almost all the \( N \) evals satisfy

\[-2|\lambda_{i,N}| = \gamma_{typ} + \mathcal{O}((\log N)^{-1/2}),\]

where \( \gamma_{typ} = -2 \int \log |d(q, p)| \, dq \, dp \) is the typical damping rate (\(|d_\infty(q, p)| = e^{-\gamma_{typ}/2} \) almost everywhere) [Sjöstrand’00, N-Schenck’08].

Is the width of the distribution really \( \mathcal{O}((\log N)^{-1/2}) \)? (OK for the solvable quantized baker’s map).
Large deviation estimates for $d_n \Rightarrow fractal upper bounds$ for the density of resonances away from $\gamma_{typ}$.

**Theorem.** [Anantharaman’08,Schenck’08]

$$\forall \alpha \geq 0, \quad \# \{ i : -2 \log |\lambda_{i,N}| \approx \gamma_{typ} + \alpha \} \leq C_{\alpha} N f(\alpha)$$

$f(\alpha) \in [0, 1] \leftrightarrow$ the rate function for $d_n$.

Solvable baker’s map: the above bound is generally not sharp.

One can also bound the decay rates using an adapted topological pressure.

**Theorem.** [Schenck’09]  *For any $\epsilon > 0$ and any large enough $N \sim \hbar^{-1}$,*

$$-2 \log |\lambda_{i,N}| \geq -2 \mathcal{P} \left( -\frac{1}{2} \log J^+ + \log |d| \right) - \epsilon$$

In some situations, the RHS is larger than $-2 \log \max |d_{\infty}|$. 
Phase space distribution of metastable states (3)

Partially open system [Asch-Lebeau’00, N-Schenck’09]: semiclassical measures associated with metastable states satisfy

\[ |d|^2 \times \tilde{\kappa} \mu = \lambda^2 \mu. \]

Such condit. invar. measures are more difficult to classify than in the fully open case.

Several numerical studies for a chaotic dielectric cavity [Wiersig, Harayama, Kim..]

Examples of Husimi measures for a partially open 3-baker.

Work in progress...