A short introduction to Loewner chains and SLE

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Foreword

The purpose of these notes is very modest. They are meant to introduce readers gently to the concepts of Loewner chains, local growth and stochastic Loewner evolutions (SLEs). These concepts have played an important role in physics and mathematics during the recent years.

A prerequisite is a minimal knowledge of complex analysis and probability theory.

The discussion is informal. There is no claim at originality. We try to give some intuition based on explicit examples. Physical applications are sometimes mentioned but never explained in detail.

There is no reference but it is easy to find more detailed and/or rigorous and/or applied and/or ... presentations with references in the available literature on the web by typing keywords.

Chapter 1

Loewner chains

There are many possible descriptions of subsets of a set. Some may look more natural than others but it is the problem at hand that decides which one is the most efficient. Growth processes in two dimensions involve time dependent subsets of the complex plane $\mathbb{C}$. Loewner chains have proved to be an invaluable tool in this context. The simplest situation is when they are used to describe families of domains. These notes deal (almost) exclusively with that case.

Loewner chains were introduced (by Loewner!) in the context of the Bieberbach conjecture, now a theorem proved by de Branges in 1985. It states that if $f(z) = z + \sum_{n\geq 2} a_n z^n$ is a holomorphic function injective in the unit disc $\mathbb{U} = \{z \in \mathbb{C}, |z| < 1\}$ then $|a_n| \leq n$ for $n \geq 2$. Bieberbach proved that $|a_2| \leq 2$ in 1912, and Loewner proved in 1923 that $|a_3| \leq 3$ using a dynamical picture of the changes of $f(\mathbb{U})$ when the a_n's change, starting from the trivial case $f(z) = z$.

1.1 Around Riemann's theorem

A *domain* $\mathbb{D}$ is a non empty connected and simply connected open set strictly included in the complex plane $\mathbb{C}$. Simple connectedness is a notion of purely topological nature which in two dimensions asserts essentially that a domain has no holes and is contractile: the domain has the same topology as a disc.

Riemann's theorem states that two domains $\mathbb{D}$ and $\mathbb{D}'$ are always conformally equivalent, i.e. there is an invertible holomorphic map $g : \mathbb{D} \mapsto \mathbb{D}'$

between them.

Riemann stated the theorem but his argument had many gaps. This was at least partly at the origin of the formidable development of functional analysis in the twentieth century but it took decades before a proof meeting modern standards was found.

Extending g to the boundary of $\mathbb{D}$ is impossible in general if the naïve notion of boundary is used, i.e. if the boundary of $\mathbb{D}$ is taken as the complement of $\mathbb{D}$ in its closure. As an example, take $\mathbb{D}$ to be the upper half plane $\mathbb{H}$ with the vertical line segment $]0, ia]$ removed and $\mathbb{D}' = \mathbb{H}$. The naïve boundary of $\mathbb{D}$ is the union of $\mathbb{R}$ and $]0, ia]$. The limits of $g(z)$ when z approaches a given point of the segment $]0, ia]$ from the left or from the right must be distinct. But another notion of boundary can be defined for which a continuous extension at the boundary is always possible. Intuitively, this more involved notion keeps track of the different sides from which a naïve boundary point can be approached. This is trivial in our simple example but the general case is involved and we shall not give a precise definition. We shall freely use the word "boundary" in the sequel, leaving to the reader the task of deciding from the context which kind of boundary we have in mind. In case when there is only one way to approach naïve boundary points the two notions coïncide.

In simple cases, the map f can be found in closed form. For instance, the upper-half plane $\mathbb{H}$ and the unit disc $\{z \in \mathbb{C}, |z| < 1\}$ centered on the origin are two domains. The conformal transformation $f(z) = i\frac{1-z}{1+z}$ maps the unit disc biholomorphically onto the upper half plane with $f(0) = i$ and $f(1) = 0$. But the general case is another matter.

The upper half plane has a three dimensional Lie group of conformal automorphisms, $PSL_2(\mathbb{R})$, that also acts on the boundary of $\mathbb{H}$. This group is made of homographic transformations $f(z) = \frac{az+b}{cz+d}$ with a, b, c, d real and $ad - bc = 1$. To specify such map we have to impose three real conditions. Hence, there is a unique automorphism – possibly followed by a transposition – that maps any triple of boundary points to any other triple of boundary points. Similarly there is unique homographic transformation that maps any pair made of a bulk point and a boundary point to another pair of bulk and boundary points. By Riemann's theorem, any domain has a Lie group of conformal automorphisms isomorphic to $PSL_2(\mathbb{R})$ and the same normalization conditions can be used.

Riemann's theorem is used repeatedly in the rest of these notes. It is the starting point of many approaches to growth phenomena in two dimensions since it allows to code the shapes of growing domains in their uniformizing conformal maps. To make the description precise one has to choose a reference domain against which the growing domains are compared. By Riemann's theorem we may choose any domain as reference domain – and depending on the geometry of the problem some choices are more convenient than others. The unit disc and the upper half plane are often used as reference domains.

1.2 Hulls

One can be a more explicit when the domain $\mathbb{D}$ differs only locally from the upper half plane $\mathbb{H}$, that is if $\mathbb{K} = \mathbb{H} \setminus \mathbb{D}$ is bounded. Such a set $\mathbb{K}$ is called a hull. The real points in the closure of $\mathbb{K}$ in $\mathbb{C}$ form a compact set which we call $\mathbb{K}_{\mathbb{R}}$. In that case, $\mathbb{H}$ is the convenient reference domain. Let $g : \mathbb{D} \mapsto \mathbb{H}$ be a conformal bijection. For $z \in \mathbb{D}$ define $g(\overline{z}) \equiv \overline{g(z)}$. If z approaches a point x on the real axis while staying within $\mathbb{D}$, $g(z)$ has a real limit which we denote by $g(x)$. It follows that g extends to a holomorphic map on the connected open set $\mathbb{D} \cup \overline{\mathbb{D}} \cup (\mathbb{R} \setminus \mathbb{K}_{\mathbb{R}}) \cup \infty$ of the Riemann sphere, which contains a neighborhood of ∞. This is an illustration of the Schwartz reflection principle. One can use the automorphism group of $\mathbb{H}$ to ensure that $g(z) = z + O(1/z)$ for large z. This is called the *hydrodynamic normalization*. It involves three conditions : g maps ∞ to ∞, has unit derivative there, and has no constant term. These three condition are real because ∞ is on the boundary of the upper half plane seen within the Riemann sphere. There is no further freedom left. Thus any property of g is an intrinsic property of $\mathbb{K}$.

We shall denote this special representative by $g_{\mathbb{K}}$. The inverse map $f_{\mathbb{K}}$ is holomorphic on the full Riemann sphere except for cut along a compact subset of $\mathbb{R}$ across which its imaginary part has a positive discontinuity (in general this is a measure) $d\mu(x)$. Away from the cut, $f_{\mathbb{K}}$ has the standard representation

$$f_{\mathbb{K}}(w) = w - \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\mu(x)}{w - x}.$$

The coefficients of the expansion of $f_{\mathbb{K}}$ at infinity are essentially the moments of μ. I particular, they are real. Each of them quantify an intrinsic property of $\mathbb{K}$. The number $C_{\mathbb{K}} \equiv \frac{1}{\pi}\int_{\mathbb{R}} d\mu(x)$ is the total mass of μ. It is positive (or 0 is $\mathbb{K}$ is empty). Note that $f_{\mathbb{K}}(w) = w - C_{\mathbb{K}}/w + \cdots$ at large w and by inverting, $g_{\mathbb{K}}(z) = z + C_{\mathbb{K}}/z + \cdots$ at large z. The coefficient $C_{\mathbb{K}}$ plays an important role. It is called the capacity of $\mathbb{K}$ seen from ∞. It's positivity is intuitively related to the fact that one *removes* a piece from $\mathbb{H}$. Capacity is trivially translation invariant ($x + \mathbb{K}$, the translate of $\mathbb{K}$ by x units along the real axis, and $\mathbb{K}$ have the same capacity) and has weight 2 under dilations ($C_{s\mathbb{K}} = s^2 C_{\mathbb{K}}$ if s is a positive scale factor). Capacity has an additive property: simple series manipulations show that if $\mathbb{K}'$ and $\mathbb{K}''$ are two hulls and $\mathbb{K} = \mathbb{K}' \cup g_{\mathbb{K}'}^{-1}(\mathbb{K}'')$ (which is another hull) then $C(\mathbb{K}) = C(\mathbb{K}') + C(\mathbb{K}'')$. .

1.3 Basic examples

Example 1 : *The semidisc.*

Maybe the simplest example is when $\mathbb{K}$ is a semidisc $\{z \in \mathbb{H}, |z - b| \leq r\}$ for a real b and real positive r. Then $g_{\mathbb{K}}(z) = z + r^2/(z - b)$. Expansion at large z shows that $C_{\mathbb{K}} = r^2$.

Example 2 : *The vertical line segment.*

In the example when $\mathbb{K}$ is the vertical line segment $]0, ia]$, one gets $g_{\mathbb{K}}(z) = \sqrt{z^2 + a^2}$, a formula by which we mean that analytic continuation of the function $z\sqrt{1 + a^2/z^2}$ were the square root is defined by its usual power series around 1 when z is large. Expansion at large z shows that $2C_{\mathbb{K}} = a^2$.

Example 3 : *The oblique line segment.*

The case when $\mathbb{K}$ is an oblique line segment $]0, ae^{i\pi b}]$ making an angle πb with respect to the real positive axis ($b \in]0, 1[$) yields

$$z = (g_{\mathbb{K}}(z) - x_+)^b (g_{\mathbb{K}}(z) - x_-)^{1-b},$$

where the real parameters $x_- < 0 < x_+$ satisfy $bx_+ + (1 - b)x_- = 0$ and $(-x_-)^b x_+^{1-b} = a$. Expansion at large z shows that $2C_{\mathbb{K}} = b(1-b)(x_+ - x_-)^2$.

The closer the line is to the real axis (i.e. the closer b is to 0 or π) and the larger a has to be to reach a given capacity.

Example 4 : *Arc of circle.*

An instructive example is when $\mathbb{K}$ is the arc $]r, re^{i\vartheta}]$ of a circle centered at 0 of radius r. Some of the following computations require to keep a precise track of the determination of the square root that appears in the formula for $g_{\mathbb{K}}$ because it is crucial for the interpretation. The map $f(w) = (w-r)/(w+r)$ sends the arc to the vertical line segment $]0, i\tan\vartheta/2]$, so that by the previous example, $w \mapsto \sqrt{f(w)^2 + \tan^2\vartheta/2}$ is a conformal map from $\mathbb{D}$ to $\mathbb{H}$. However, this map sends ∞ to $1/(\cos\vartheta/2)$, not to ∞. To get the hydrodynamic normalization, we have to compose with an appropriate automorphism of $\mathbb{H}$. This yields

$$g_{\mathbb{K}}(w) = r\frac{-(2-\cos^2\vartheta/2)\cos\vartheta/2\sqrt{\left(\frac{z-r}{z+r}\right)^2 + \tan^2\vartheta/2} + 2 - 3\cos^2\vartheta/2}{\cos\vartheta/2\sqrt{\left(\frac{z-r}{z+r}\right)^2 + \tan^2\vartheta/2} - 1},$$

whose expansion at ∞ starts like $g_{\mathbb{K}}(w) = w + (1-\cos^4\vartheta/2)r^2/w + O(1/w^2)$. Hence the capacity is $C_{\mathbb{K}} = (1-\cos^4\vartheta/2)r^2$.

The tip of the arc, $re^{i\vartheta}$ is mapped to $(3\cos^2\vartheta/2 - 2)r$ by $g_{\mathbb{K}}$. One checks that

$$(g_{\mathbb{K}}(w) - g_{\mathbb{K}}(re^{i\vartheta}))\frac{\partial g_{\mathbb{K}}(w)}{\partial\vartheta} = 2r^2\sin\vartheta/2\ \cos^3\vartheta/2,$$

which is w-independent.

Moreover $\lim_{w\mapsto r^-} g_{\mathbb{K}}(w) = r(1-2\sin\vartheta/2 - \sin^2\vartheta/2)$ and $\lim_{w\mapsto r^+} g_{\mathbb{K}}(w) = r(1+2\sin\vartheta/2 - \sin^2\vartheta/2)$. The behavior of $g_{\mathbb{K}}$ when $\vartheta \mapsto \pi^-$ is interesting. In this limit, $\mathbb{K}$ becomes a semicircle. Let $\tilde{\mathbb{K}} = \{w \in \mathbb{H}, |w| \leq r\}$ be the corresponding semidisc. The points w inside $\tilde{\mathbb{K}}$ are cut away from ∞ when $\vartheta \mapsto \pi^-$, and one checks that $\lim_{\vartheta\mapsto\pi^-} g_{\mathbb{K}}(w) = -2r$ for these points, i.e. they are swallowed in the limit. However, the points $\{w \in \mathbb{H}, |w| > r\}$ are mapped to $\lim_{\vartheta\mapsto\pi^-} g_{\mathbb{K}}(w) = w + r^2/w = g_{\tilde{\mathbb{K}}}(w)$.

1.4 Iteration of conformal maps

With Riemann's theorem at our disposal, we can start to encode growth processes. Suppose than the initial domain is the upper half plane and that a small amount of matter is removed at each time step (so that in fact it is the lower half plane that grows). At time step n, a certain $\mathbb{K}_n$ has been removed from $\mathbb{H}$. Let $g_n \equiv g_{\mathbb{K}_n}$ denote the corresponding map and f_n its inverse. Then $g_n(\mathbb{K}_{n+1} \setminus \mathbb{K}_n)$ describes a small amount of matter removed to $\mathbb{H}$. If $g_n(\mathbb{K}_{n+1} \setminus \mathbb{K}_n)$ has typical size s and is located in the neighborhood of point x on the real axis, $\mathbb{K}_{n+1} \setminus \mathbb{K}_n$, which is what is really removed at time $n+1$ has typical size $s|f'_n(x)|$.

Example 5 : *Simple iteration.*

Choose a small number ε. Let b_n, $n > 0$ be an independent sequence drawn from some chosen probability distribution. At time step $n+1$ take $g_n(\mathbb{K}_{n+1} \setminus \mathbb{K}_n)$ to be the semidisc $\{z \in \mathbb{H}, |z - b_{n+1}||f'_n(b_{n+1})| \leq \varepsilon\}$, so that

$$g_{n+1}(z) = g_n(z) + \frac{\varepsilon^2}{|f'_n(b_{n+1})|^2(g_n(z) - b_{n+1})}.$$

This defines a random growth process were at each time step a small semidisc-like grain of matter of size $\sim \varepsilon$ is removed. Despite its simplicity, little is known (at least to the author) about this process.

Many other (probabilistic or deterministic) rules can be invented, but the resulting processes are mostly impossible to study analytically at the moment.

1.5 Continuous time growth processes

Our aim is to motivate the introduction of Loewner chains.

If $\mathbb{K}$ is not simply a semidisc, but an union of well-separated small semidiscs of radii r_α centered at b_α, a moment of thought leads to realize that

$$g_{\mathbb{K}}(z) \sim z + \sum_\alpha \frac{r_\alpha^2}{z - b_\alpha}.$$

The large z expansion yields $C_{\mathbb{K}} \sim \sum_\alpha r_\alpha^2$, a positive number as expected.

Taking a naïve limit, one gets that if ε is a small positive number, $\nu(x)$ is a nonnegative function on $\mathbb{R}$ and $\mathbb{K} = \{z = x + iy \in \mathbb{H}, y \leq \varepsilon\nu(x)\}$ then

$$g_{\mathbb{K}}(z) \sim z + \frac{\varepsilon}{\pi} \int_{\mathbb{R}} \frac{\nu(u)du}{z - u}.$$

Indeed, using that, if $\nu(x) \neq 0$, $\lim_{\varepsilon \mapsto 0^+} \mathfrak{Im}\,(x + i\varepsilon\nu(x) - u)^{-1} = \pi\delta(u - x)$ one checks that $\mathfrak{Im}\,\frac{1}{\pi}\int_{\mathbb{R}} \frac{\nu(u)du}{x+i\varepsilon\nu(x)-u} \sim -\nu(x)$ so that to first order in ε $g_{\mathbb{K}}(z)$ is real when z is on the boundary of $\mathbb{K}$. Even more generally, one could replace the positive measure $\nu(u)du$ by any positive measure $d\rho(u)$. A naïve large z expansion, certainly valid if the function ν (or more generally the measure $d\rho$) has compact support and finite mass, gives $C_{\mathbb{K}} \sim \frac{\varepsilon}{\pi}\int_{\mathbb{R}} \nu(u)du$ (more generally $C_{\mathbb{K}} \sim \frac{\varepsilon}{\pi}\rho(\mathbb{R})$, again a a positive number.

Now think about a continuous time growth process for which $\mathbb{K}_t$ has been removed from $\mathbb{H}$ at time t. Let $g_t \equiv g_{\mathbb{K}_t}$ denote the corresponding map and f_t its inverse. Fix t and a small positive ε. Then $g_t(\mathbb{K}_{t+\varepsilon}\backslash\mathbb{K}_t)$ describes a small amount of matter removed to $\mathbb{H}$. We could take as a definition of continuous time growth that the associated map $g_{t+\varepsilon} \circ f_t$ is described by a nonnegative function $\nu_t(u)$ or more generally a positive measure $d\rho_t(u)$ as above. Taking the limit $\varepsilon \mapsto 0^+$ leads to

$$\frac{\partial g_t(z)}{\partial t} = \frac{1}{\pi}\int_{\mathbb{R}} \frac{d\rho_t(u)}{g_t(z) - u}. \qquad (1.1)$$

Such an evolution equation is called a Loewner chain with reference domain $\mathbb{H}$. The analogous equations with reference domain the unit disc can be obtained straightforwardly by the same arguments. The large z expansion yields

$$\frac{dC_{\mathbb{K}_t}}{dt} = \frac{1}{\pi}\rho_t(\mathbb{R}).$$

So if hulls are constructed little by little by a growth process, the positivity of capacity is obvious.

In principle, if the family of measures ρ_t is given, one can solve for $g_t(z)$ with the initial condition $g_0(z) = z$. Again, ρ_t can be random or deterministic. We should note that Loewner chains are in some sense kinematic equations that give a general framework to encode growth processes. But in a real dynamical problem ρ_t has to be specified. It may depend explicitly on g_t. For instance $d\rho_t(u) = |f_t'(u)|^{-2}du$ is related to Laplacian growth, though the unit disc geometry is the relevant one in that case. The exponent -2, which we already interpreted for discrete iteration, ensures that the size of $\mathbb{K}_t$ grows linearly with time. But other exponents between 0 and -2 are interesting too.

1.6 Geometric interpretation

One can give a the following geometric interpretation of Loewner chains. Set $g_t(z) \equiv z_t$, view z_t as the position of a fluid particle as time goes by, and suppose for simplicity that $d\rho_t(u) = \nu_t(u)du$ so that the Loewner chain becomes

$$\frac{dz_t}{dt} = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\nu_t(u)du}{z_t - u}.$$

Hence $\frac{1}{\pi}\int_{\mathbb{R}} \frac{\nu_t(u)du}{z-u}$ plays the role of a time dependent holomorphic vector field on the manifold with boundary $\mathbb{H}$. At point $z = x + i0^+$ i.e. close to the the real axis (the boundary of $\mathbb{H}$) this vector field has imaginary part $-\nu(x)$, so that when x is away from the support of ρ_t, (that is, when $\nu_t(.) = 0$ in a neighborhood of x), the vector field is real, i.e. tangent to the boundary. However, if x is on the support of ρ_t the vector field has a finite negative imaginary part, which means that some fluid particles that started inside $\mathbb{H}$ can be swallowed by the boundary. In fact $\mathbb{K}_t$ is nothing but the set of fluid particles which where in $\mathbb{H}$ at $t = 0$ but have hit the boundary before time t.

The reader is urged to review the examples 1-4 in this light. For the semidisc case, take r as time, either with $b = 0$ or with $b = r$. For the case of line segments, take a constant b and use a as time. For the arc of circle, using ϑ as time, with special care in the limit $\vartheta \mapsto \pi^-$. It is instructive to compute the measure ρ_t in each case and to check that the above interpretation of $\mathbb{K}_t$ is correct.

Another, more abstract, geometric interpretation is also possible. Let N_- be the group of series of the form $z + \sum_{m\leq -1} g_m z^{m+1}$ with real coefficients and convergent for large z (the domain of convergence may depend on the series, so N_- is made of "germs", and is in fact the group of germs of holomorphic functions fixing ∞ and with derivative 1 at ∞). In the same spirit, let O_∞ be the space of germs of holomorphic functions at infinity. We let N_- act on O_∞ by composition, $\gamma_g \cdot F \equiv F \circ g$. Observe that $\gamma_{g_1 \circ g_2} = \gamma_{g_2} \cdot \gamma_{g_1}$ so this is an anti-representation.

Note that the g_t's of a Loewner chain with bounded $\mathbb{K}_t$ belong to N_-. If $F \in O_\infty$ and if z is large enough, $F(z)$ is well defined as well as $F(z_t)$ for small t (where the meaning of small may depend on z and F) and

$$\frac{dF(z_t)}{dt} = \frac{1}{\pi}\int_{\mathbb{R}} \frac{d\rho_t(u)}{z_t - u}\frac{\partial F}{\partial z}(z_t),$$

which can be rewritten

$$\frac{d}{dt}(\gamma_{g_t} \cdot F) = \gamma_{g_t} \cdot (\nu_t \cdot F)$$

where $\nu_t(z) \equiv \frac{1}{\pi}\int_{\mathbb{R}} \frac{d\rho_t(u)}{z-u}\frac{\partial}{\partial z}$ is a germ of vector field.

So the Loewner chain equation can be viewed as a flow on N_-

$$\frac{d}{dt}\gamma_{g_t} = \gamma_{g_t} \cdot \nu_t.$$

The group N_- has an interesting representation theory, related to that of the Virasoro algebra, which can be used as a probe for this flow.

1.7 Local growth

Suppose that as time goes by the measures ρ_s are δ-peaks of height $\pi a_s/2$ (the factor 2 is purely historical) at position ξ_s. In the upper half plane reference geometry, the growth process will be described by an equation of the type

$$\frac{\partial g_s(z)}{\partial s} = \frac{2a_s}{g_s(z) - \xi_s}.$$

Note that examples 2-4 fall in this category. The formula was given for example 4 if $s = \vartheta$ and the other cases lead to simple computations left to the reader.

If one is interested only in the growth of the hull, but not in the way the evolution is parameterized, one can make change the time variable without arm. The statement that ξ_s changes quickly or slowly makes sense only compared with the changes in a_s. For instance, suppose that the function a_s vanishes in some interval, while ξ_s keeps on changing so that it has a different value at the beginning and at the end of the interval. During that interval g_s has not changed but when a_s starts moving again, the place at which the hull resumes growth can be far from the place where it was growing before the pause. This is a limiting case of what happens when variations of ξ_s are large with respect to those of a_s. This means that if, at s_0, ξ_s starts to move very fast with respect to a_s, the growth takes place very near $\mathbb{K}_{s_0}$ or the real axis. This conclusion is supported by example 3.

We also infer that to have *local growth*, i.e. to have the position where the hull grows vary continuously, we need to impose that ξ_s stops if a_s does.

To make this statement precise, it is convenient to go to a special time parameterization. The capacity of the hull at time s is $C_{\mathbb{K}_s} = 2\int_0^s ds'\, a_{s'}$, a non-decreasing function of s. Define $t = \int_0^s ds'\, a_{s'}$, take t to be the new time variable and by abuse of notation write ξ_t for $\xi_{s(t)}$, $\mathbb{K}_t$ for $\mathbb{K}_{s(t)}$ and so on. Then by construction $C_{\mathbb{K}_t} = 2t$ and the equation reads

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \xi_t} \tag{1.2}$$

We take as a definition of *local growth* that ξ_t is continuous function of t. The function ξ_t is often called the *driving function* of the Loewner evolution. It is sometimes convenient to normalize ξ_t by $\xi_0 = 0$ or what amounts to the same to impose that the hull starts growing from point 0.

A broad class of growing hulls that can be described by such an equation is given by continuous simple curves started on the boundary of $\mathbb{H}$ and staying in $\mathbb{H}$ thereafter. Let $\gamma_{[0,\infty]}$ be a parameterized simple continuous curve from 0 to ∞ in $\mathbb{H}$ and assume that the capacity parameterizations has been chosen, so that $\mathbb{K}_t \equiv \gamma_{]0,t]}$ is a hull with capacity 2t. When ε is small, $\mathbb{K}_{\varepsilon,t} \equiv g_t(\gamma_{]t,t+\varepsilon]})$ is a tiny piece of a curve. The support of the discontinuity measure $d\rho_{f_{\varepsilon,t}}$ is small and becomes a point when ε goes to 0. Measures supported at a point are δ functions, so there is a point ξ_t such that, as a measure, $d\rho_{f_{\varepsilon,t}}/dx \sim 2\varepsilon\delta(x - \xi_t)$ as $\varepsilon \to 0^+$.

For a general local Loewner growth process, one defines $\gamma_t = f_t(\xi_t + i0^+) \equiv \lim_{\varepsilon \mapsto 0^+} f_t(\xi_t + i\varepsilon)$ (remember f_t is the inverse map of g_t). We shall often use the shorthand notation $\gamma_t = f_t(\xi_t)$. The set $\gamma_{]0,t]} \equiv \cup_{s \in]0,t]} \gamma_s$ is called *the trace* of the growth process. If the hull is a simple curve, the notation is consistent. Whether the trace is a curve (simple or not) in general is highly non obvious, but this will be the case for all examples in these notes, though proving it can be a formidable task.

At time t, growth takes place at point ξ_t in the g_t plane i.e. at point γ_t in the original "physical" plane. Thus it is tempting to conclude that $\mathbb{K}_t$ coincides with $\gamma_{]0,t]}$. Though this picture works nicely for examples 2-3, it is slightly too naïve and fails in the example 4 when the trace, which is an arc of circle closes to a semicircle and the corresponding semidisc completes the hull.

For a given z with $\Im m z \geq 0$ and $z \neq \xi_0$, the local existence and uniqueness of solutions to eq.(1.2) is granted by general theorems on ordinary differential equations, but problems may arise if a time τ_z (depending on z in general) exists for which $g_{\tau_z}(z) = \xi_{\tau_z}$. One possibility is to declare $g_t(z)$ undefined for $t \geq \tau_z$. But it is often the case that, as suggested by examples 2-3, the two limits $\lim_{x \mapsto \xi^{\pm}_{\tau_z}} g_t \circ f_{\tau_z}(x)$ exist, allowing to think that after τ_z, $g_t(z)$ has split in two real trajectories.

There is regularity criterion on the function $\xi_{\cdot}$ that guaranties that if $x \neq \xi_0$ is real, τ_x is infinite. It is sufficient that for each t,

$$\lim_{s \mapsto t^-} \sup_{t' \in [s,t[} \frac{|\xi_t - \xi_{t'}|}{|t-t'|^{1/2}} < 4. \tag{1.3}$$

To prove this criterion, it is convenient to consider $X_t \equiv g_t(x) - \xi_t$, a continuous function which satisfies the integral equation $X_t = x - \xi_t + \int_0^t \frac{2ds}{X_s}$. As this implies that $\xi_\tau - \xi_t = X_t - X_\tau + \int_t^\tau \frac{2ds}{X_s}$, we can see $\xi_{\cdot}$ as a functional of $X_{\cdot}$. The task is to control its behavior if X_t has a given sign, say positive, on $[0, \tau[$ and vanishes at τ. It is clear that the two terms in $X_t + \int_t^\tau \frac{2ds}{X_s}$ vary in opposite directions, in that the faster X_t goes to 0 , the slower is the vanishing of $\int_t^\tau \frac{2ds}{X_s}$ at $t = \tau$. So the mildest behavior of the sum as t goes to τ is when the two terms have a similar behavior. A detailed analysis requires some care, but a quick and dirty way to retrieve the criterion is to impose that the two terms be equal, which gives $X_t = 2\sqrt{\tau - t}$ hence $\xi_\tau - \xi_t = 4\sqrt{\tau - t}$ as announced.

Example 6 : *Square root driving term.*

The Loewner equation when $\xi_\tau - \xi_t = 4\alpha\sqrt{\tau - t}$ can be solved in closed form for any α though the formulæ are cumbersome. We normalize ξ_t so that $\xi_0 = 0$, i.e. take $\xi_t = 4\alpha(\sqrt{\tau} - \sqrt{\tau - t})$. By left-right symmetry, we can assume that $\alpha \geq 0$. For $\alpha \in [1, +\infty[$ it is convenient to set $\alpha \equiv \cosh\eta$, $\eta \in [0, +\infty[$. One parameterizes time as

$$\frac{2e^{-\eta\coth\eta}\sinh\eta}{\sin(2\vartheta\sinh\eta)} \frac{(\sin(\vartheta e^{\eta}))^{(\coth\eta+1)/2}}{(\sin(\vartheta e^{-\eta}))^{(\coth\eta-1)/2}} = \sqrt{\frac{\tau - t}{\tau}},$$

with $\vartheta \in [0, \pi e^{-\eta}]$. As a function of ϑ, the hull builds the curve

$$\left\{2\sqrt{\tau}\left(e^{-\eta} - \frac{2\sinh\eta\sin(\vartheta e^{-\eta})}{\sin(2\vartheta\sinh\eta)} e^{i\vartheta e^{\eta}}\right)\right\}_{\vartheta \in [0,\pi e^{-\eta}]}.$$

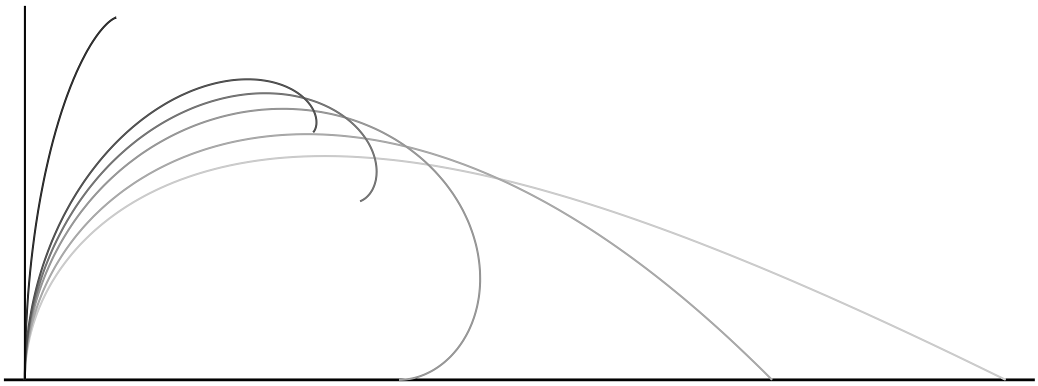

Figure 1.1: The hull at time τ for $\alpha = 0, 1, 2, 3, 4, 5, 6$

For $\vartheta = \pi e^{-\eta}$ the curve closes a whole domain, just as in the arc of circle example 4, which in fact is the special case $\alpha = 3\sqrt{2}$.

For $\alpha \in [0, 1[$ it is convenient to set $\alpha \equiv \cos\varphi$, $\varphi \in]0, \pi/2]$. The formulæ can be obtained by analytic continuation $\eta \mapsto i\varphi$, this time with a parameter $\vartheta \in [0, \infty]$. The hulls remain simple curves even for $\vartheta = \infty$.

Fig.1.7 illustrates the different behaviors.

The very same criterion on the behavior of the function $\xi_.$ is also sufficient for to ensure that the hull $\mathbb{K}_t$ is a simple continuous curve, say $\{\gamma_s, s \in]0, t]\}$, and $\gamma_t = f_t(\xi_t)$, i.e. that our naïve expectation $\mathbb{K}_t = \cup_{s\in]0,t]} f_s(\xi_s)$ is fulfilled.

The two properties – "$g_t(x))$ for real x does not hit ξ_t" and "the hull is a simple curve"– are in fact equivalent. The intuitive reason is the following. The fact that $g_t(x))$ for real x hits ξ_t at some time τ is the sign that at time τ the hull "swallows a whole piece of $\mathbb{H}$". The previous example illustrates this relationship when the hull hits the real axis. But from the point of view of iteration, if $s \geq 0$ is fixed, it is obvious that when $t \geq 0$ varies the function $\tilde{g}_{t,s}(z) \equiv g_{t+s} \circ f_s(z + \xi_s) - \xi_s$ satisfies the Loewner equation (1.2) with driving function $\tilde{\xi}_t \equiv \xi_{t+s} - \xi_s$. So if the driving function $\tilde{\xi}_t \equiv \xi_{t+s} - \xi_s$ leads to a hull hitting the real axis, the driving function ξ_t leads to a hull hitting itself or the real axis. This discussion also explains why, if the trace is a continuous curve, it can have double points but no crossings.

Chapter 2

Stochastic Loewner evolutions

Stochastic Loewner evolutions were introduced by Schramm in 1999 as a general framework to study random curves satisfying certain properties. His specific interest was to prove that loop erased random walks (in short lerw's, the precise definition is irrelevant here) on a two dimensional lattice have a conformally invariant continuum limit. Schramm observed that these walks have on the lattice the so-called domain Markov property (to be defined below) a property that can that can be rephrased in the continuum. Though he was not able at that time to prove the existence of a conformally invariant limit of lerw's, he recognized that conformal invariance and the domain Markov property brought together would have remarkable consequences, and was able to prove that the probability measures on random curves in the continuum satisfying at the same time conformal invariance and the domain Markov property formed a one parameter family. Crucial to the proof and the explicit description of these measures was the idea of viewing curves as hulls and to use Loewner evolutions. That in this context the most useful description of a curve is by encoding it into a growth process via a Loewner chain is at first sight very surprising and may explain why physicists who had understood the importance of conformal invariance to study many examples of random curves in the early 1980's failed to "produce Schramm's argument before Schramm".

The general idea is to impose properties relating different members in a family of probability measures on continuous curves without crossings, but possibly with multiple points. Let us note that curves here are considered modulo reparameterizations, but not simply as subsets of the plane. For simple curves, this would essentially make no difference, but curves with

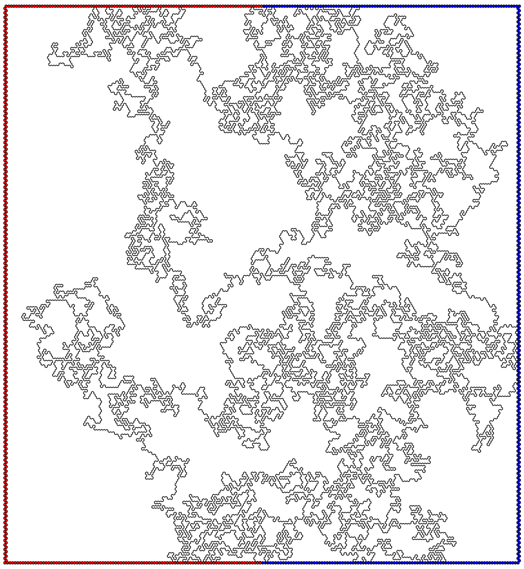

Figure 2.1: A percolation sample

multiple points require more care.

In the discrete setting, it is a fact that interfaces on appropriate lattices are simple curves, so why bother to deal with non simple curves ? The answer is that even if at the scale of the lattice spacing the interface is simple, when one tries to take a continuum limit by looking at a macroscopic scale while taking a smaller and smaller lattice spacing, a curve that makes a large excursion and then comes back close to itself, say a few lattice spacings away, has a double point from the macroscopic viewpoint. While in some models –like lerw's, Schramm's initial motivation– the interface remain simple when the lattice spacing gets smaller, some other models –like percolation– clearly exhibit multiple points in the continuum limit. This is clearly seen on samples, see Fig.2.

In the following three sections, we suppose that we are given a family of probability measures $\{\mathbf{P}_{\mathbb{D},a,b}\}$ indexed by triples consisting of a domain $\mathbb{D}$ and two distinct boundary points a, b of $\mathbb{D}$. For a given triple $(\mathbb{D}, a, b)$, $\mathbf{P}_{\mathbb{D},a,b}$ is a a measure on $\Omega_{\mathbb{D},a,b}$, the set of continuous curves without crossings within $\overline{\mathbb{D}}$ –the union of $\mathbb{D}$ and its boundary (in the refined sense alluded too in section 1.1)– joining a to b (it is understood that a and b are not multiple points).

First, we want do define what it means for the family $\{\mathbf{P}_{\mathbb{D},a,b}\}$ to be conformally invariant and to have the domain Markov property.

2.1 Conformal invariance

By Riemann's theorem, if $(\mathbb{D}, a, b)$ and $(\mathbb{D}, a', b')$ are any two triples, there is a conformal map $g : \mathbb{D} \mapsto \mathbb{D}'$ such that $g(a) = a'$ and $g(b) = b'$. It is clear that g induces a bijection, which we call $\check{g}$, from $\Omega_{\mathbb{D},a,b}$ to $\Omega_{\mathbb{D}',a',b'}$. Conformal invariance of the family $\{\mathbf{P}_{\mathbb{D},a,b}\}$ is the statement that $\check{g}$ is measurable and the image measure $\mathbf{P}_{\mathbb{D},a,b} \circ \check{g}^{-1}$ coincides with $\mathbf{P}_{\mathbb{D}',a',b'}$, i.e. if C' is a measurable subset of $\Omega_{\mathbb{D}',a',b'}$ then $\check{g}^{-1}(C')$ is a measurable subset of $\Omega_{\mathbb{D},a,b}$ and $\mathbf{P}_{\mathbb{D},a,b}(\check{g}^{-1}(C')) = \mathbf{P}_{\mathbb{D}',a',b'}(C')$.

Conformal invariance by itself is a rather weak constraint. Indeed, suppose that a probability $\mathbf{P}_{\mathbb{D}_0,a_0,b_0}$ on $\Omega_{\mathbb{D}_0,a_0,b_0}$ has been defined for a single triple $\mathbb{D}_0, a_0, b_0$ and that it is invariant under the conformal transformations of $\mathbb{D}_0$ fixing a_0 and b_0. Such transformations form a group with one real parameter. Then the direct image $\mathbf{P}_{\mathbb{D}_0,a_0,b_0}$ by any conformal transformation g will define unambiguously $\mathbf{P}_{g(\mathbb{D}_0),g(a_0),g(b_0)}$. By the Riemann mapping theorem, this defines $\mathbf{P}_{\mathbb{D},a,b}$ for any triple, and the resulting family of probabilities is clearly conformally invariant.

To get a more rigid situation, one has to impose another constraint on the family $\{\mathbf{P}_{\mathbb{D},a,b}\}$. Schramm translated in the continuum a property that holds for loop erased random walks in the discrete setting : the domain Markov property, to which we turn our attention now.

Before doing so, let us remark that this strategy is rather typical. If continuous curves without crossings are replaced by general hulls joining a to b in $\mathbb{D}$ the notion of domain Markov property does not make sense but another one, restriction, turns out to be fruitful and allow for another complete classification. We shall have little to say about these nice "restriction measures" in the sequel.

2.2 Domain Markov property

Fix a triple $(\mathbb{D}, a, b)$ and consider an element $\gamma \in \Omega_{\mathbb{D},a,b}$. If a real continuous parameter along γ is given and s is any intermediate value of the parameter, the past and the future of s split γ in two (not necessarily disjoint) curves without crossings. The curve corresponding to the past of s starts at a and is called an initial segment of γ. The curve corresponding to the future of s ends at b and is called a final segment of γ. The final segment starts at some point $c \in \mathbb{D}$ which is also the end of the initial segment. We use the notation $\gamma_{]a,c]}$ for such an initial segment with point c included and $\gamma_{]c,b]}$ for the final segment. Beware that the notation is a bit ambiguous, because of possible multiple points on γ.

Several curves γ' share the same initial segment $\gamma_{]a,c]}$, and the discussion that follows focuses on the question : if an initial segment is given, what is the distribution of the final segment?

Making sense of this question is not so obvious. First, there should be enough measurable sets in $\Omega_{\mathbb{D},a,b}$. We shall for a while assume that this is so. But even in that case, the event "γ' starts exactly with$\gamma_{]a,c]}$" is more than likely to occur with probability 0. Vaguely, what may have a nontrivial probability is the event "γ' has an initial segment that is close (in some quantified sense) to $\gamma_{]a,c]}$". Probabilists have invented so called conditional expectations and regular conditional probabilities just to deal with that kind of situations. Starting from $\mathbf{P}_{\mathbb{D},a,b}$ this allows to define new probability measures, denoted $\mathbf{P}_{\mathbb{D},a,b}(\quad |\gamma_{]a,c]})$, read "conditional probability given the initial segment $\gamma_{]a,c]}$", that can be manipulated just as conditional probabilities when the state space is discrete[1].

The set of points in $\mathbb{D}$ that cannot be joined to b by a continuous curve in $\mathbb{D}$ without hitting the initial segment form a set that we call a hull[2] and denote by $\mathbb{K}_c$. This notation is again slightly ambiguous. Note that $\mathbb{D} \setminus \mathbb{K}_c$ is again a domain. If the initial segment is $\gamma_{]a,c]}$, the final segment starts at c and never enters inside $\mathbb{K}_c$. So the support of the conditional probability $\mathbf{P}_{\mathbb{D},a,b}(\quad |\gamma_{]a,c]})$ is included in $\Omega_{\mathbb{D}\setminus\mathbb{K}_c,c,b}$. But on this set we have another probability measure, namely $\mathbf{P}_{\mathbb{D}\setminus\mathbb{K}_c,c,b}$, and the two can be compared.

We say that a set $\{\gamma_{]a,c]}\}$ of curves in $\mathbb{D}$ without crossings starting at a is a set of distinct representatives if any curve in $\Omega_{\mathbb{D},a,b}$ has exactly one of its initial segments in $\{\gamma_{]a,c]}\}$. For instance, for the triple $(\mathbb{H}, 0, \infty)$, the

[1] There is a small price to pay, however. For instance, the definition of this conditional probability may fail or be ambiguous for certain $\gamma_{]a,c]}$ but these nasty initial segments form altogether a set of probability 0 for $\mathbf{P}_{\mathbb{D},a,b}$.

[2] If $(\mathbb{D}, a, b) = (\mathbb{H}, 0, \infty)$, this is consistent with our initial definition, and with the new definition, conformal maps send hulls to hulls

initial segments whose associated hull has capacity t form a set of distinct representatives. Intuitively, to get the expectation of a random variable on $\Omega_{\mathbb{D},a,b}$, one can compute its conditional expectation on $\gamma_{]a,c]}$, and then integrate over $\gamma_{]a,c]}$ in a system of distinct representatives.

The family $\{\mathbf{P}_{\mathbb{D},a,b}\}$ is said to have the *domain Markov property* if, for any triple $(\mathbb{D},a,b)$, $\mathbf{P}_{\mathbb{D},a,b}(\quad|\gamma_{]a,c]}) = \mathbf{P}_{\mathbb{D}\setminus\mathbb{K}_c,c,b}$ except maybe for a set of initial segments whose intersection with any system of distinct representatives is of measure 0 for $\mathbf{P}_{\mathbb{D},a,b}$.

This expression of the domain Markov property is more intuitive on the lattice in the discrete setting –because the interfaces are simple curves and because conditional probabilities have a much simpler definition– and it holds in may examples. It is vaguely related to the notion of locality in physics.

2.3 Schramm's argument

Our aim is to explore the interplay between conformal invariance and the domain Markov property of the family $\{\mathbf{P}_{\mathbb{D},a,b}\}$.

First, by conformal invariance, we may concentrate on the triple $(\mathbb{H},0,\infty)$. We choose a parameterization of curves in $\Omega_{\mathbb{H},0,\infty}$ in such a way that the hull $\mathbb{K}_t \equiv \mathbb{K}_{\gamma_t}$ associated with the initial segment $\gamma_{]0,t]} \equiv \gamma_{]0,\gamma_t]}$ of $\gamma \in \Omega_{\mathbb{H},0,\infty}$ has capacity $2t$. Because of the underlying continuous curve γ, the growth of $\mathbb{K}_t$ is local, and the associated g_t satisfies a Loewner equation $\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z)-\xi_t}$ for some continuous function ξ_t. The probability $\mathbf{P}_{\mathbb{H},0,\infty}$ on $\Omega_{\mathbb{H},0,\infty}$ induces a a random process on the set of initial segments $\gamma_{]0,t]}$, hence on the set of hulls $\mathbb{K}_t$, and on the set of continuous functions ξ_t.

Our next aim is to derive consequences for the stochastic process ξ_t of the domain Markov property and conformal invariance.

First for fixed $(\mathbb{H},0,\infty)$ there is a remnant of conformal invariance : dilations. Hence for $\lambda > 0$, the hull $\frac{1}{\lambda}\mathbb{K}_{\lambda^2 t}$ must have the same distribution as a $\mathbb{K}_t$. The corresponding Loewner map is $\frac{1}{\lambda}g_{\lambda^2 t}(\lambda z)$, whose driving function is $\frac{1}{\lambda}\xi_{\lambda^2 t}$. Hence the processes ξ_t and $\frac{1}{\lambda}\xi_{\lambda^2 t}$ have the same law. We say that ξ_t has dimension $1/2$

Given K_t, the domain Markov property states that $\gamma_{]t,\infty]}$ is distributed according to $\mathbf{P}_{\mathbb{D}\setminus\mathbb{K}_t,\gamma_t,\infty}$. The conformal transformation $g_t(z) - \xi_t$ maps $\mathbb{D}\setminus\mathbb{K}_t$ to $\mathbb{H}$, γ_t to 0 and ∞ to ∞. By conformal invariance, $g_t(\gamma_{]t,\infty]}) - \xi_t$

is distributed according to $\mathbf{P}_{\mathbb{H},0,\infty}$. In particular for $s \geq 0$ $g_t(\gamma_{]t,t+s]}) - \xi_t$ has the same distribution as a $\gamma_{]0,s]}$ hence is independent of $\gamma_{]0,t]}$. But the Loewner map for $g_t(\gamma_{]t,t+s]}) - \xi_t$ is $g_{s+t} \circ f_t(z + xi_t) - \xi_t$ (remember f_t is the inverse of g_t), whose driving function is $\xi_{t+s} - \xi_t$. We infer that the random function $\xi_.$ is such that for any $t, s \geq 0$, $\xi_{t+s} - \xi_t$ is independent of $\{xi_{t'}\}$, $t' \in [0, t]$ and distributed like a ξ_s.

To resume our knowledge, the random process $\xi_.$ has continuous samples, independent identically distributed increments and dimension 1/2. By a deep general result, a random process with continuous samples and independent identically distributed increments is of the form $\sqrt{\kappa}B_t + \rho t$ for some nonnegative κ and some real ρ. Obviously it has dimension 1/2 if and only if $\rho = 0$.

To conclude, Schramm's argument shows that if a family of probabilities $\{\mathbf{P}_{\mathbb{D},a,b}\}$ on curves without crossing indexed by triples $(\mathbb{D}, a, b)$ is conformally invariant and has the domain Markov property, the law induced by $\mathbf{P}_{\mathbb{H},0,\infty}$ on initial hulls of capacity 2t by is described by a stochastic Loewner evolution

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \sqrt{\kappa}B_t} \tag{2.1}$$

for some $\kappa \geq 0$ and some normalized Brownian motion B_t.

A priori, this does not show that each κ is realized via some family $\{\mathbf{P}_{\mathbb{D},a,b}\}$ (because the Loewner evolution deals wit hulls, not with curves).

2.4 Basic properties

The first important property is a kind of converse to Schramm's result. If $\kappa \geq 0$ is a real number, and B_t a continuous realization of a normalized Brownian motion, a deep theorem states that the trace associated to the stochastic Loewner evolution eq.(2.1) is almost surely a continuous curve joining 0 to ∞. This curve is simple and stays in $\mathbb{H}$ if $\kappa \in [0, 4]$, has double points and hits the real axis if $\kappa \in]4, 8[$ and is spacefilling if $\kappa \in [8, +\infty[$.

At the time Schramm introduced stochastic Loewner evolutions, this very hard theorem was not known (he contributed to prove it later).

As explained before, a continuous trace cannot have crossings. Thus for any $\kappa \geq 0$, the stochastic Loewner evolution defines a probability measure $\mathbf{P}_\kappa$ on continuous curves without crossings joining 0 to ∞ in $\mathbb{H}$. This measure is scale invariant. Hence, for each κ, conformal transformations can be used to define in a consistent way a family of probabilities $\{\mathbf{P}^{\kappa}_{\mathbb{D},a,b}\}$. This family is trivially conformally invariant, and it is easy to check that is satisfies the domain Markov property.

This finishes the complete classification.

Taking the existence of a curve for granted, the change of behavior from simple curves to curves with double points at $\kappa = 4$ can be understood as follows. First, the necessary condition (negation of eq.(1.3) for the existence of multiple points is fulfilled for all values of κ, though in some kind of marginal way, for if $\xi_t = \sqrt{\kappa} B_t$ where B_t is a normalized Brownian motion, the law of the iterated logarithm states that, with probability one

$$\lim_{s \mapsto t^-} \sup_{t' \in [s,t[} \frac{|\xi_t - \xi_{t'}|}{|t-t'|^{1/2} \log\log |t-t'|^{-1}} = \sqrt{2\kappa}.$$

So the stochastic Loewner source is wilder by a $\log\log |t-t'|^{-1}$ than the criterion. The fact that for $\kappa \leq 4$ the Loewner trace is a simple curve shows that, as should be expected, the criterion is only necessary, but not sufficient. Intuitively, Brownian motion is more singular than necessary, but for $\kappa \leq 4$ with too little correlation time to behave consistently for long enough periods to produce multiple points.

This fact is related to another well studied question : recurrence of Brownian motion. If space dimension d is 1, Brownian motion passes infinitely many times at any point, if $d = 2$, it passes infinitely many times in the any neighborhood of any point, but not exactly at any given point, and if $d \geq 3$, it has a nonzero probability to remain at a given finite distance of any point. So dimension 2 is somehow a marginal case. Now let R_t be the norm of a d-dimensional Brownian motion. Assume $R_0 > 0$. One can show using stochastic calculus that $W_t \equiv -R_t + \frac{d-1}{2}\int_0^t \frac{ds}{R_s}$ is a standard 1-dimensional Brownian motion. In this equation, d appears as a an explicit parameter, and one can reverse the logic : given a standard 1-dimensional Brownian motion W_t what are the properties of R_t, called the d-dimensional Bessel process mathematics. Setting $\kappa = 4/(d-1)$ one sees that $X_t \equiv \sqrt{\kappa}(R_t + W_t)$ satisfies the equation $\frac{dX_t}{dt} = \frac{2}{X_t - \sqrt{\kappa} W_t}$ so first, indeed from W_t on can retrieve R_t by solving a differential equation and second, the Bessel process is essentially a stochastic Loewner evolutions but looking only at the boundary of $\mathbb{H}$. For general d, the Bessel processes behave with respect to visits to 0 just like the recurrence properties of

Brownian motion for integer d suggest: the d-dimensional Bessel process hits the origin infinitely many times if $d < 2$, but never if $d \geq 2$. Equivalently, if $\kappa \leq 4$, $X_t - \sqrt{\kappa} W_t$ never vanishes, but vanishes infinitely many times if $\kappa > 4$. But we already know that the vanishing of $X_t - \sqrt{\kappa} W_t$ is the sign that the growing curve hits itself or the real axis.

Another very hard result is the fractal dimension : the measures $\mathbf{P}^{\kappa}_{\mathbb{D},a,b}$ is concentrated on curves with fractal dimension $\min\{1 + \kappa/8, 2\}$.

Two additional properties have been used to constraint further the situation.

The first one is locality. Let $\mathbb{L}$ be a hull in $\mathbb{D}$ bounded away from a and b. To each curve in $\Omega_{\mathbb{D},a,b}$ we can associate its smallest initial segment that hits the boundary of $\mathbb{L}$ (we take this initial segment to be the curve itself if it never hits $\mathbb{L}$. These initial segments form a system Σ of distinct representatives both in $\Omega_{\mathbb{D},a,b}$ and in $\Omega_{\mathbb{D}\setminus\mathbb{L},a,b}$. Thus both $\mathbf{P}_{\mathbb{D},a,b}$ and $\mathbf{P}_{\mathbb{D}\setminus\mathbb{L},a,b}$ induce a probability measure on Σ. The property of locality is the statement that these two measures coincide. In a more mundane way, if $\mathbb{L}$ is a hull in $\mathbb{D}$ bounded away from a and b, the distribution of curves up to the first hitting of $\mathbb{L}$ are the same in $\mathbb{D}$ and in $\mathbb{D} \setminus \mathbb{L}$. Stochastic calculus can be used to show that the family $\{\mathbf{P}^{\kappa=6}_{\mathbb{D},a,b}\}$ is the only one to have the locality property. Let us note that it is no surprise that a value of κ satisfying locality is > 4. Indeed, if $\kappa \leq 4$, the traces are simple curves that do not hit the boundary. Then no trace touches $\mathbb{L}$ for $\mathbf{P}_{\mathbb{D}\setminus\mathbb{L},a,b}$, but hitting $\mathbb{L}$ for $\mathbf{P}_{\mathbb{D},a,b}$ has a finite probability if $\mathbb{L}$ is nontrivial, so that the supports of the two probability measures induced on Σ are not the same.

On the lattice, percolation is modelled by coloring each site with one of two colors, independently of the other sites. The associated interfaces have obviously the locality property, so the only candidate if percolation has a conformally invariant continuum limit is the family with $\kappa = 6$. That percolation has a conformally invariant continuum limit has indeed been proved.

The second one is the restriction property. Again, let $\mathbb{L}$ be a hull in $\mathbb{D}$ bounded away from a and b. Consider the subset $\Gamma_{\mathbb{L}}$ of $\Omega_{\mathbb{D},a,b}$ made of curves that do not hit $\mathbb{L}$, and the associated conditional probability $\mathbf{P}_{\mathbb{D},a,b}(\quad|\Gamma_{\mathbb{L}})$. Note that $\Gamma_{\mathbb{L}}$ is a subset of $\Omega_{\mathbb{D}\setminus\mathbb{L},a,b}$. The restriction property is the statement that $\mathbf{P}_{\mathbb{D},a,b}(\quad|\Gamma_{\mathbb{L}}) = \mathbf{P}_{\mathbb{D}\setminus\mathbb{L},a,b}$. Stochastic calculus can be used to show that the family $\{\mathbf{P}^{\kappa=8/3}_{\mathbb{D},a,b}\}$ is the only one to have the restriction property. This restriction property was alluded to before. Indeed, properly defined restriction measures form a one parameter family of measures on hulls, which intersect the SLE family at $\kappa = 8/3$. Let us note that the filling of a Brownian excursion is another example of restriction measure. Again, it is no surprise that a value of κ satisfying restriction is ≤ 4. Indeed, if $\kappa > 4$, the probability of hitting $\mathbb{L}$ under $\mathbf{P}_{\mathbb{D}\setminus\mathbb{L},a,b}$ would be nonzero, so that $\Gamma_{\mathbb{L}}$ would not have full measure in $\Omega_{\mathbb{D}\setminus\mathbb{L},a,b}$.

On the lattice, the weight of a self avoiding walk is given in the plane, and then the same weight is used for this curve in any domain that contains it. So self avoiding walks on the lattice have the restriction property. So the only candidate if self avoiding walks have a conformally invariant continuum limit is the family with $\kappa = 8/3$. But this time a proof of the existence of a continuum limit of self avoiding walks is still to come.

Recently, two important conjectures on SLE have been proven.

One of them is reversibility. The treatment of random curves by a Loewner evolution is quite asymmetric by definition. However interfaces between two points in physics (i.e. in statistical mechanics models) quite generally make no difference between the two ends. So it was conjectured very early that interfaces generated by an SLE process were reversible. One difficulty is with the parameterization. Take an SLE sample in $\mathbb{H}$ from 0 to infinity, parameterize it with capacity. Apply the transformation $z \mapsto -1/z$ and parameterize the inversed sample with capacity. Now any point on the curve has two parameters attached to it. One of the troubles is that the relationship between the two parameters is extremely wild. Anyway, reversibility is now a theorem.

The second one is duality. Take an SLE_κ sample with $\kappa > 4$ and look at the boundary of $\mathbb{K}_t$. This is simple curve, and one can expect that its distribution is conformally invariant in some sense. So it is natural to ask if and how it fits in the SLE framework. It was conjectured by physicists that it is related in some sense to an $SLE_{16/\kappa}$, and that in particular it has dimension $1 + 2/\kappa$. Though this is correct, the precise recent theorem that gives an explicit description involves nontrivial extensions of SLE where the driving function is $\sqrt{16/\kappa}B_t$ plus some rather complicated drift terms.