

# Introduction to Scattering Amplitudes

## Lecture 1: QCD and the Spinor-Helicity Formalism

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### Abstract

Scattering amplitudes take surprisingly simple forms in theories such as quantum chromodynamics (QCD) and general relativity. This simplicity indicates deep symmetry. Recently, it has become possible to explain some of this symmetry. I will describe these insights and show how to derive amplitudes efficiently and elegantly. Key new ideas involve using complexified momentum, exploring singular behavior, and seeking clues in so-called twistor geometry. Complete amplitudes can be produced recursively. This streamlined approach is being applied in searches for new physics in high-energy particle colliders.

Lecture 1: Color quantum numbers can be set aside. Helicity amplitudes have elegant expressions in terms of spinors. MHV (maximally helicity violating) amplitudes are the simplest.

## 1 Introduction

The theme of this course is the study of scattering amplitudes through their *simplicity* and *singularities*.

When written as functions of well chosen variables, formulas for scattering amplitudes take simpler forms than one would naively expect from thinking of sums of Feynman diagrams. This simplicity is clearest in supersymmetric Yang-Mills theory, but extends to pure Yang-Mills, the Standard Model, and even (super)gravity. Simplicity motivates *why* and *how* we compute them: new structures demand new insights, and lead to new computational methods.

Explicit computations of amplitudes are important for experimental studies, notably in hadron colliders; and more formally, for exploring deeper structures in field theories such as Yang-Mills and supergravity.

## Precision calculations at hadron colliders

Hadron colliders such as the Tevatron and especially LHC have very large QCD backgrounds. In order to observe signals of new physics, both background and signal must be computed to high precision (on the order of 1%). This typically means computing to next-to-leading order (NLO) in the strong coupling constant, and in some cases to next-to-next-to-leading order as well. One- and two-loop computations are thus of particular interest. These higher-order computations also have the effect of reducing renormalization scale dependence.

The QCD Factorization Theorem states that thanks to asymptotic freedom, an infrared-safe, collinear-safe observable can be expressed as a convolution of parton distribution functions with hard scattering kernels.

$$\sum_{a,b} \int_0^1 dx_1 dx_2 f_{a/h_1}(x_1, \mu_f) H_{ab}(Q; Q^2/\mu^2, \mu_f/\mu, \alpha_s(\mu)) f_{b/h_2}(x_2, \mu_f)$$

The parton distribution function  $f_{a/h_1}(x_1, \mu_f)$  is the probability density of finding parton  $a$  of momentum fraction  $x_1$  in proton  $h_1$  at energy scale  $\mu_f$ . Therefore it is meaningful to compute cross sections in perturbative QCD.

## $\mathcal{N} = 4$ supersymmetric Yang-Mills theory

This is a very special theory. It is conformal, with maximal supersymmetry in four dimensions. It is integrable in the planar limit. It is involved in various dualities: most famously, AdS/CFT; relevant to this course, a duality with twistor string theory; and internal dualities among Wilson loops, amplitudes, and correlation functions that are still being discovered today. Some practitioners like to refer to it as “the harmonic oscillator of the 21st century.” We will come back to this subject in Lecture 3, but for now I just want to touch upon a couple of motivations for pushing the boundaries of amplitude calculations.

For years, the “cusp anomalous dimension” or “soft anomalous dimension”  $f(\lambda)$  has been a computational target. It is the scaling of twist-2 operators in the limit of large spin,

$$\Delta(\text{Tr}[ZD^S Z]) - S = f(\lambda) \log S + \mathcal{O}(S^0).$$

Weak coupling calculations have been done from QCD pdf’s and gluon amplitudes, while a strong coupling expansion comes from AdS/CFT, integrability considerations and an all-loop Bethe ansatz. The number has been matched through 4 loops, giving checks on proposals for integrable structures.

Another inspiration has been the “BDS ansatz” (Bern, Dixon, Smirnov), a conjecture for iterative structure,

$$\log \left( \frac{A_n}{A_n^{\text{tree}}} \right) = \text{Div}_n + \frac{f(\lambda)}{4} a_1(k_1, \dots, k_n) + h(\lambda) + nk(\lambda)$$

The formula was conjectured based on the expected divergences of the amplitude from soft & collinear limits. By now we know that it *fails* for  $n > 5$ , and it has

been instructive to see why it could be satisfied for  $n \geq 5$  (new symmetries!) and how it fails for  $n > 5$ , since the difference, called the “remainder function” has been predicted in the  $n \rightarrow \infty$  limit from strong-coupling considerations. Specifically, AdS/CFT predicts that the amplitude can be computed from the classical action of the string worldsheet whose boundary is a certain polygon. This correspondence extends to weak coupling as well.

## Gravity amplitudes

Amplitudes in Einstein gravity or supergravity are simple as well, though we might not expect it from looking at the Lagrangian. Starting from a string theory analysis by Kawai, Lewellen and Tye (KLT) relating closed string amplitudes and open string amplitudes, one can take the field theory limit and relate graviton and gauge field amplitudes. These KLT relations are valid through at least two loops in  $\mathcal{N} = 8$  supergravity.

One basic question in gravity theories is, could  $\mathcal{N} = 8$  supergravity actually be finite? Like  $\mathcal{N} = 4$  SYM, it is special by virtue of having maximal supersymmetry and no additional field content. Indeed, supersymmetry plays a role in suppressing expected divergences, but supersymmetry arguments alone can only eliminate divergences through a certain number of loops. Depending on the sophistication of the argument, the number starts at 3 and (so far) goes up to 9. No one has performed a 9-loop computation, but thanks to this motivation, 4-loop amplitudes have been computed and further computations are underway. Along the way, new identities have been discovered enhancing our understanding of amplitudes in general or in  $\mathcal{N} = 4$  SYM. One reason to think it is possible that the theory is ultimately finite is that its relation to  $\mathcal{N} = 4$  SYM is even stronger than we currently understand.

## 2 QCD at tree level

Now we begin studying amplitudes in detail. To look at concrete examples, we choose QCD as a theory amenable to new techniques. However, the spinor-helicity formalism can be generalized to include other field content. Massless fields are straightforward, while spinors for massive fields are somewhat less clean and are less widely used. The extension to supersymmetric Yang-Mills theory is easy and we will use it later.

Our field content is the gluon, transforming in the adjoint representation of the gauge group  $SU(N)$ , and quarks and antiquarks of assorted flavors, transforming in the fundamental or antifundamental representation. The Feynman rules are given in Figure 1.

**Recommended reading:** Most of this introductory lecture follows [1] closely. I refer you to those lectures for additional explanations and a complete list of references. I am also using material from [2], especially in using Weyl spinors rather than Dirac spinors. A new lecture write-up [3] appeared just last week.

$$\begin{aligned}
& \text{Three-gluon vertex: } = -g f^{abc} (\eta_{\nu\rho} (p-q)_\mu + \eta_{\rho\mu} (q-k)_\nu + \eta_{\mu\nu} (k-p)_\rho) \\
& \text{Four-gluon vertex: } = -ig^2 f^{abe} f^{cde} (\eta_{\lambda\nu} \eta_{\mu\rho} - \eta_{\lambda\rho} \eta_{\mu\nu}) \\
& \quad - ig^2 f^{ace} f^{bde} (\eta_{\lambda\mu} \eta_{\nu\rho} - \eta_{\lambda\rho} \eta_{\mu\nu}) \\
& \quad - ig^2 f^{ade} f^{cbe} (\eta_{\lambda\nu} \eta_{\mu\rho} - \eta_{\lambda\mu} \eta_{\rho\nu}) \\
& \text{Quark-gluon vertex: } = -ig \gamma_\mu T_{ij}^a \\
& \text{Gluon propagator: } = -i \frac{\delta^{ab} \eta_{\mu\nu}}{p^2} \\
& \text{Ghost propagator: } = \frac{i \delta^{ij}}{p}
\end{aligned}$$

Figure 1: Feynman rules for QCD in Lorentz (Feynman) gauge with massless quarks, omitting ghosts.

It covers color ordering and the spinor-helicity formalism, with many examples of computing amplitudes in QCD and QED, and including an introduction to BCFW recursion relations.

## 2.1 Color ordering

The  $SU(N)$  color algebra is generated by the  $N \times N$  traceless hermitian matrices  $T^a$ , with the color index  $a$  taking values from 1 to  $N^2 - 1$ . They are normalized by  $\text{Tr}(T^a T^b) = \delta^{ab}$ . The structure constants  $f^{abc}$  are defined by  $[T^a, T^b] = i\sqrt{2} f^{abc} T^c$ , from which it follows that

$$f^{abc} = -\frac{i}{\sqrt{2}} (\text{Tr}(T^a T^b T^c) - \text{Tr}(T^a T^c T^b)). \quad (1)$$

Gluon propagators conserve color through the factor  $\delta^{ab}$ . The traces in the  $f^{abc}$  at the vertices can be merged by the Fierz identity,

$$\sum_a (T^a)_i^{\bar{j}} (T^a)_k^{\bar{\ell}} = \delta_i^{\bar{\ell}} \delta_k^{\bar{j}} - \frac{1}{N} \delta_i^{\bar{j}} \delta_k^{\bar{\ell}} \quad (2)$$

It follows that at tree level, all color factors combine to form a single trace factor for each term. (One can check that the terms with  $1/N$  all cancel among themselves; this is guaranteed by the fact that this term would be absent if the

gauge group were  $U(N)$  rather than  $SU(N)$ , but the auxiliary photon field does not couple to gluons anyway.)

The color decomposition for gluon amplitudes at tree level is:

$$\begin{aligned} \mathcal{A}_n^{\text{tree}}(\{a_i, p_i, \epsilon_i\}) = & \quad (3) \\ g^{n-2} \sum_{\sigma \in S_n / \mathbb{Z}_n} \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \dots T^{a_{\sigma(n)}}) A(p_{\sigma(1)}, \epsilon_{\sigma(1)}, \dots, p_{\sigma(n)}, \epsilon_{\sigma(n)}). \end{aligned}$$

The sum is over all permutations of the gluon labels, with a quotient by  $\mathbb{Z}_n$  because the trace is cyclically invariant, so all these permutations can be combined in the same term. The function  $A(p_i, \epsilon_i)$  of kinematic arguments only is called the ‘‘color-ordered partial amplitude.’’ Once we have performed the color decomposition, we will refer to this function simply as the ‘‘amplitude’’ of the process. For tree amplitudes involving quarks, analogous formulas can be derived where instead of a trace, the string of matrices will be terminated by the specific  $T^a$ ’s in the quark-quark-gluon vertices.

One-loop amplitudes of gluons have double-trace terms as well as single-trace terms. Their color decomposition is:

$$\begin{aligned} \mathcal{A}_n^{1\text{-loop}}(\{a_i, p_i, \epsilon_i\}) = & \quad (4) \\ g^n \left[ \sum_{\sigma \in S_n / \mathbb{Z}_n} N \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \dots T^{a_{\sigma(n)}}) A_{n;1}(p_{\sigma(1)}, \epsilon_{\sigma(1)}, \dots, p_{\sigma(n)}, \epsilon_{\sigma(n)}) \right. \\ & + \sum_{c=2}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n / S_{n;c}} \text{Tr}(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(c-1)}}) \text{Tr}(T^{a_{\sigma(c)}} \dots T^{a_{\sigma(n)}}) \\ & \left. \times A_{n;c}(p_{\sigma(1)}, \epsilon_{\sigma(1)}, \dots, p_{\sigma(n)}, \epsilon_{\sigma(n)}) \right]. \end{aligned}$$

Here, the partial amplitude  $A_{n;1}$  multiplying the single-trace term is called the leading-color partial amplitude, and the  $A_{n;c}$  are called subleading-color partial amplitudes. There is a relation among these partial amplitudes, so that in fact it suffices to compute the leading-color partial amplitudes. The  $A_{n;c}$  are then fixed by the identity

$$A_{n;c}(1, 2, \dots, c-1; c, c+1, \dots, n) = (-1)^n \sum_{\sigma \in COP\{\alpha\}\{\beta\}} A_{n;1}(\sigma), \quad (5)$$

where  $\{\alpha\}$  is the reverse-ordered set  $\{c-1, c-2, \dots, 2, 1\}$ ,  $\{\beta\}$  is the ordered set  $\{c, c+1, \dots, n\}$ , and  $COP$  denotes the cyclically-ordered permutations of  $\{1, \dots, n\}$  preserving the cyclic orderings of  $\{\alpha\}$  and  $\{\beta\}$ .

Having carried out the color decomposition, we now turn our attention to the (leading-color) partial amplitudes. The upshot is that we only need to consider *planar* diagrams for a given *cyclic* ordering of gluons. There are suitably defined ‘‘color-ordered Feynman rules’’ generating these partial amplitudes, given in Figure 2.

$$\begin{aligned}
 & \text{Diagram 1: } \rho, q, k, \mu, p, \nu \quad = \frac{i}{\sqrt{2}}(\eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu + \eta_{\mu\nu}(k-p)_\rho) \\
 & \text{Diagram 2: } \mu, \nu, \lambda, \rho \quad = i\eta_{\mu\rho}\eta_{\nu\lambda} - \frac{i}{2}(\eta_{\mu\nu}\eta_{\rho\lambda} + \eta_{\mu\lambda}\eta_{\nu\rho}) \\
 & \text{Diagram 3: } \mu, p, p' \quad = \frac{i}{\sqrt{2}}\gamma_\mu \quad \text{Diagram 4: } \mu, \nu, p \quad = -i\frac{\eta_{\mu\nu}}{p^2} \\
 & \text{Diagram 5: } \mu, p, p' \quad = -\frac{i}{\sqrt{2}}\gamma_\mu \quad \text{Diagram 6: } p, p' \quad = \frac{i}{p'}
 \end{aligned}$$

Figure 2: Color-ordered Feynman rules in QCD. Momenta are directed outwards from the cubic vertex.

## 2.2 Spinor-helicity formalism

The color-ordered amplitudes are functions of the (null) momentum 4-vectors  $p_i$  and the polarization vectors  $\epsilon_i$ . In the spinor-helicity formalism, these numbers will be exchanged for spinors and helicity labels. The motivation is that there is too much redundancy among momenta and polarizations. Given a momentum vector  $p_i$  for an external gluon, we know that the polarization vectors must be transverse and therefore satisfy  $\epsilon_i \cdot p_i = 0$ , and moreover that the shift  $\epsilon_i \rightarrow \epsilon_i + w p_i$  for constant  $w$  is a gauge transformation that must leave the amplitude invariant. However, there is no natural choice of  $\epsilon_i$  given these constraints. By expressing the momentum in terms of spinors, we will improve this situation. This section follows the exposition in [2].

The Lorentz group is locally isomorphic to  $SL(2) \times SL(2)$ , whose finite-dimensional representations are labeled by  $(p, q)$  taking integer or half-integer values. The positive-chirality spinor representation is  $(1/2, 0)$ , while the negative-chirality spinor representation is  $(0, 1/2)$ . We write  $\lambda_a$  for a positive-chirality spinor and  $\tilde{\lambda}_{\dot{a}}$  for a negative-chirality spinor, where the labels  $a$  and  $\dot{a}$  take values 1, 2.

The vector representation of  $SO(3, 1)$  is the  $(1/2, 1/2)$  representation of  $SL(2) \times SL(2)$ , so a momentum vector should be viewed as an object with one each of the positive and negative chirality spinor labels. To see this map explicitly, let us work in the chiral (Weyl) basis of gamma matrices,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad (6)$$

where  $\sigma^\mu = (1, \vec{\sigma})$  and  $\bar{\sigma}^\mu = (-1, \vec{\sigma})$ . Given a Lorentz vector  $p_\mu$ , define a 2x2 matrix

$$p_{a\dot{a}} = \sigma_{a\dot{a}}^\mu p_\mu = p_0 + \vec{\sigma} \cdot \vec{p}. \quad (7)$$

One can see that  $p_\mu p^\mu = \det(p_{a\dot{a}})$ . Thus a null momentum vector  $p$  is mapped to a 2x2 matrix whose rank is strictly less than 2 and can therefore be written in the form

$$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}. \quad (8)$$

For real-valued momentum vectors in  $+- --$  signature, the spinors  $\lambda_a$  and  $\tilde{\lambda}_{\dot{a}}$  are actually complex-valued, but they are complex conjugates of each other, up to a sign. Later on, we will find it indispensable to work with complex-valued momentum vectors, for which  $\lambda_a$  and  $\tilde{\lambda}_{\dot{a}}$  become completely independent.

Notice that the equation (8) does not give unique values for the spinors. It is always possible to exchange a constant factor between them. In real Minkowski space, since the spinors should be complex conjugates, this factor can only be a complex phase. The choice of a specific pair of spinors for a given momentum vector is equivalent to the choice of a wavefunction for a spin 1/2 particle of that same momentum. (This is the starting point in the exposition of spinor-helicity in [1].)

Explicit forms for the spinors associated to a null vector  $(p_0, p_1, p_2, p_3)$  are

$$\lambda_a = \frac{e^{i\theta}}{\sqrt{p_0 - p_3}} \begin{pmatrix} p_1 - ip_2 \\ p_0 - p_3 \end{pmatrix}, \quad \tilde{\lambda}_{\dot{a}} = \frac{e^{-i\theta}}{\sqrt{p_0 - p_3}} \begin{pmatrix} p_1 + ip_2 & p_0 - p_3 \end{pmatrix}, \quad (9)$$

where  $e^{i\theta}$  is the freely chosen phase factor. The spinor indices are raised and lowered with the epsilon tensors  $\epsilon_{ab}, \epsilon_{\dot{a}\dot{b}}$  and their inverses  $\epsilon^{ab}, \epsilon^{\dot{a}\dot{b}}$ .

### 2.3 Spinor products and notation

Two spinors of the same chirality can be contracted with the epsilon tensors. We use different shapes of brackets for the two chiralities, and define antisymmetric spinor products as

$$\epsilon^{ab}\lambda_a\mu_b \equiv \langle\lambda\mu\rangle = -\langle\mu\lambda\rangle, \quad (10)$$

$$\epsilon^{\dot{a}\dot{b}}\tilde{\lambda}_{\dot{a}}\tilde{\mu}_{\dot{b}} \equiv [\tilde{\mu}\tilde{\lambda}] = -[\tilde{\lambda}\tilde{\mu}]. \quad (11)$$

Just as we had  $p \cdot p = \det(p_{a\dot{a}})$ , it is easy to see that  $p \cdot q = \frac{1}{2}\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}p_{a\dot{a}}q_{b\dot{b}}$ . If both these vectors are null, we find

$$p \cdot q = \frac{1}{2}\langle\lambda\mu\rangle[\tilde{\mu}\tilde{\lambda}]. \quad (12)$$

In the literature, it is very common to find 4-component Dirac spinors rather than the 2-component Weyl spinors. We summarize the correspondence and shorthand below. The subscripts on the Dirac spinors indicate helicity.

Weyl shorthand	Weyl spinor	Dirac shorthand	Dirac spinor pos. energy	Dirac spinor neg. energy
$ i\rangle$	$\lambda_a(p_i)$	$ i^+\rangle$	$u_+(p_i)$	$v_-(p_i)$
$ i]$	$\tilde{\lambda}^{\dot{a}}(p_i)$	$ i^-\rangle$	$\frac{u_-(p_i)}{2}$	$\frac{v_+(p_i)}{2}$
$\langle i $	$\lambda^a(p_i)$	$\langle i^- $	$\frac{u_-(p_i)}{2}$	$\frac{v_+(p_i)}{2}$
$[i $	$\tilde{\lambda}_{\dot{a}}(p_i)$	$\langle i^+ $	$u_+(p_i)$	$v_-(p_i)$

It is also important to be aware that there are two commonly used conventions for the square-bracket spinor product (for negative chiralities), which differ by a sign. Here we do our best to follow the more traditional ‘‘QCD’’ conventions, where

$$\langle ij\rangle[ji] = 2p_i \cdot p_j = (p_i + p_j)^2 \equiv s_{ij}. \quad (13)$$

The opposite sign is used in [2] and many ‘‘twistor-inspired’’ papers that followed.

More complicated contractions of spinor indices are expressed by expanded spinor products. Notice that

$$\not{p}_i = |i\rangle[i| + |i]\langle i|. \quad (14)$$



When any Lorentz vector is sandwiched between spinors, it is redundant to write the slash, although it is common to do so. Therefore it should be understood that

$$\langle i|P|j\rangle = \lambda_a(p_i)P_{bb}\tilde{\lambda}_{\dot{a}}(p_j)\epsilon^{ab}\epsilon^{\dot{a}\dot{b}} \quad (15)$$

$$= [j|P|i]. \quad (16)$$

Chirality implies that a product such as  $\langle i|P|j\rangle$  does not exist; in Dirac spinor terminology,  $\langle i^-|P|j^+\rangle = 0$ . Here,  $P$  is an arbitrary Lorentz vector. In amplitude calculations, we will typically find  $P$  that are the sum of external momenta. Since the external legs are cyclically ordered, we define for convenience

$$P_{i,j} = p_i + p_{i+1} + \cdots + p_j \quad (17)$$

where indices are taken modulo  $n$ , the number of legs. Then, for example, our notation implies that

$$\langle k|P_{i,j}|\ell\rangle = \sum_{r=i}^j \langle ar\rangle [rb]. \quad (18)$$

Before coming back to the subject of polarization vectors and amplitudes, let us introduce one useful spinor identity, the Schouten identity:

$$0 = \langle ij\rangle \langle k\ell\rangle + \langle ik\rangle \langle \ell j\rangle + \langle i\ell\rangle \langle jk\rangle. \quad (19)$$

The Schouten identity follows from the fact that the spinors live in a 2-dimensional space.

## 2.4 Polarization vectors

Now that we have expressed the gluon momenta in terms of spinors, we can write the polarization vectors in a natural form for each of the two helicity states. For a gluon of momentum  $p_{a\dot{a}} = \lambda_a\tilde{\lambda}_{\dot{a}}$ , the polarization vectors are

$$\epsilon_{a\dot{a}}^- = -\sqrt{2}\frac{\lambda_a\tilde{\mu}_{\dot{a}}}{[\tilde{\lambda}\tilde{\mu}]}, \quad \epsilon_{a\dot{a}}^+ = -\sqrt{2}\frac{\mu_a\tilde{\lambda}_{\dot{a}}}{\langle\mu\lambda\rangle}, \quad (20)$$

where  $\mu$  and  $\tilde{\mu}$  are arbitrary reference spinors, as long as the denominators do not vanish. Since they are directly proportional to the spinors, it is clear that the transverse condition  $\epsilon \cdot p = 0$  is satisfied. It is also easy to see that the freedom to choose  $\mu$  and  $\tilde{\mu}$  is the freedom of gauge.

Thus the polarization vectors satisfy the following relations,

$$\epsilon_{a\dot{a}}^\pm \lambda^a \tilde{\lambda}^{\dot{a}} = 0, \quad (21)$$

$$\epsilon_{a\dot{a}}^+ \epsilon^{+a\dot{a}} = 0, \quad (22)$$

$$\epsilon_{a\dot{a}}^- \epsilon^{-a\dot{a}} = 0, \quad (23)$$

$$\epsilon_{a\dot{a}}^+ \epsilon^{-a\dot{a}} = -1. \quad (24)$$

## 2.5 Helicity amplitudes

With the expressions for polarization vectors, we can show that the simplest classes of helicity amplitudes of gluons vanish,

$$\begin{aligned} A(1^+, 2^+, 3^+, \dots, n^+) &= 0, \\ A(1^-, 2^+, 3^+, \dots, n^+) &= 0. \end{aligned}$$

Because cyclic relabelings are not distinct, the second equation covers any configuration with  $(n - 1)$  gluons of positive helicity and one of negative helicity.

To see that these amplitudes vanish, notice that all Lorentz indices must be contracted in the final expression. An  $n$ -point tree amplitude has at most  $n - 2$  momentum vectors in each term, from the cubic interaction vertices. That means that at least two of the polarization vectors must be contracted with each other. We can make all such contractions vanish by choosing the reference spinors wisely. If  $\mu_i = \mu_j$ , then  $\epsilon_i^+ \cdot \epsilon_j^+ = 0$ . So we choose all the  $\mu$  for positive helicities to be identical. The contraction  $\epsilon_1^- \cdot \epsilon_j^+$  vanishes if  $\tilde{\mu}_1 = \tilde{\lambda}_j$  or  $\mu_j = \lambda_1$ . We can make this choice of  $\mu_j$  for all the positive helicities in the second amplitude. Similar arguments show that  $A(1_{\bar{q}}^-, 2_q^+, 3^+, \dots, n^+) = 0$ .

Thus, the nonvanishing amplitudes start when at least two gluons have helicity opposite to the rest. These are called Maximally Helicity Violating (MHV) amplitudes.

Naturally, we have parity-conjugate vanishing relations for amplitudes with only negative-helicity gluons or a single positive-helicity gluon. Parity conjugation is one of several useful identities satisfied by helicity amplitudes.

Reflection:

$$A(1, 2, \dots, n) = (-1)^n A(n, \dots, 2, 1) \quad (25)$$

Parity conjugation:

$$A(1^{h_1}, 2^{h_2}, \dots, n^{h_n}) = (-1)^n (A(1^{-h_1}, 2^{-h_2}, \dots, n^{-h_n}))|_{(\cdot, \cdot) \leftrightarrow [\cdot, \cdot]} \quad (26)$$

Photon decoupling identity:

$$0 = A^{\text{tree}}(1, 2, 3, \dots, n) + A^{\text{tree}}(2, 1, 3, \dots, n) + A^{\text{tree}}(2, 3, 1, \dots, n) + \dots + A^{\text{tree}}(2, 3, \dots, 1, n) \quad (27)$$

This identity is derived by decoupling the non-existent photon, which would be present if the gauge group were  $U(N)$  instead of  $SU(N)$ . Then we could put the identity matrix into the trace in the color decomposition formula (4), whose form is unchanged. But there is no photon in QCD, so this color structure multiplies a vanishing expression.

Finally, the scaling of polarization vectors gives a scaling property of the full amplitude. For each particle labeled by  $i$ ,

$$\left( \lambda_i^a \frac{\partial}{\partial \lambda_i^a} - \tilde{\lambda}_i^{\dot{a}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{a}}} \right) A(\lambda_i, \tilde{\lambda}_i, h_i) = -2h_i A(\lambda_i, \tilde{\lambda}_i, h_i). \quad (28)$$

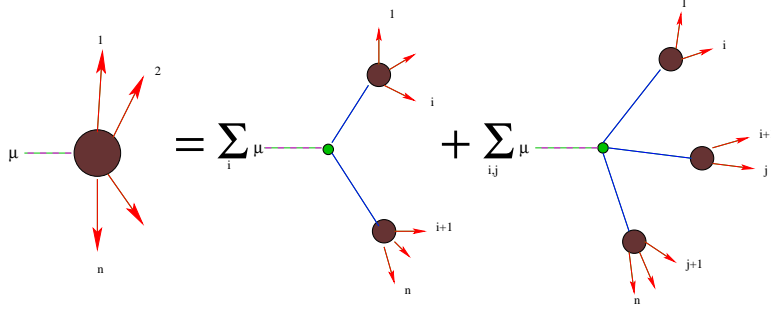


Figure 3: The Berends-Giele recursion relation.

As we think about computing amplitudes, we start for small values of  $n$ . For  $n = 4$  and  $n = 5$ , every nonvanishing amplitude is MHV or conjugate-MHV. Moreover, we can use the photon decoupling identity to limit our computations to amplitudes where the negative helicities are cyclically adjacent, i.e.  $A(-, -, +, +)$  and  $A(-, -, +, +, +)$ . Through judicious choices of reference spinors for the polarization vectors, we can drastically reduce the number of diagrams to compute and carry out the computation by hand without too much trouble. But as the number of legs increases, we will find recursive techniques immensely helpful.

## 2.6 Recursion for off-shell currents (Berends-Giele)

The Berends-Giele recursion for currents [4] generates gluon amplitudes by taking a single external leg off shell. Define  $J_\mu(1, 2, \dots, n)$  as the sum of Feynman diagrams where gluons  $1, \dots, n$  are on shell but there is one additional off-shell gluon with the uncontracted vector index  $\mu$ . The current can be constructed recursively by noticing that the gluon labeled by  $\mu$  must be attached to either a cubic or a quartic vertex, and in either case, the vertex is contracted with similar currents involving fewer legs. See Figure 3.

$$J^\mu(1, \dots, n) = \frac{-i}{P_{1,n}^2} \left[ \sum_{i=1}^{n-1} V_3^{\mu\nu\rho}(P_{1,i}, P_{i+1,n}) J_\nu(1, \dots, i) J_\rho(i+1, \dots, n) + \sum_{j=i+1}^{n-1} \sum_{i=1}^{n-2} V_4^{\mu\nu\rho\sigma} J_\nu(1, \dots, i) J_\rho(i+1, \dots, j) J_\sigma(j+1, \dots, n) \right]$$

The off-shell current satisfies the current conservation identity,

$$P_{1,\dots,n}^\mu \cdot J_\mu(1, \dots, n) = 0. \quad (29)$$

To construct the  $(n+1)$ -point gluon amplitude from the current  $J_\mu(1, \dots, n)$ , first amputate the propagator by multiplying by  $iP_{1,n}^2$ . Then, contract with

$\epsilon_{n+1}^\mu$ , the polarization vector of either helicity. Finally, take the limit  $p_{n+1}^2 = P_{1,n}^2 \rightarrow 0$ .

The algorithm is unsurpassed for its numerical power. Analytically, closed-form expressions are available for the simplest helicity configurations:

$$J^\mu(1^+, 2^+, \dots, n^+) = \frac{\langle q | \sigma^\mu P_{1,n} | q \rangle}{\sqrt{2} \langle q1 \rangle \langle 12 \rangle \cdots \langle n-1, n \rangle \langle nq \rangle}, \quad (30)$$

$$J^\mu(1^-, 2^+, \dots, n^+) = \frac{\langle q | \sigma^\mu P_{2,n} | q \rangle}{\sqrt{2} \langle 12 \rangle \cdots \langle n-1, n \rangle \langle n1 \rangle} \sum_{m=3}^n \frac{\langle 1m \rangle \langle 1 | P_{1,m} | m \rangle}{P_{1,m-1}^2 P_{1,m}^2} \quad (31)$$

These formulas were constructed from an ansatz that could be checked to satisfy the recursion.

From these results, we can confirm that amplitudes with all positive or one negative helicity vanish. More importantly, we can prove the formula conjectured by Parke and Taylor for MHV amplitudes [5]. If the negative-helicity gluons are labeled by  $j$  and  $k$ , then the amplitude is

$$A(1^+, \dots, j^-, \dots, k^-, \dots, n^+) = i \frac{\langle jk \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}. \quad (32)$$

With Berends-Giele recursion, complete analytic results for gluon amplitudes were given up through  $n = 7$ , and also the complete next-to-MHV series  $A(1^-, 2^-, 3^-, 4^+, \dots, n^+)$  where the negative-helicity gluons are cyclically adjacent.

**Exercise:** By parity conjugation of the Parke-Taylor formula for 4 gluons, we can see that

$$i \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} = i \frac{[34]^4}{[12] [23] [34] [41]}.$$

If we didn't know that this expression represented an amplitude, the equality would not be obvious. Prove it using spinor identities, given momentum conservation  $p_1 + p_2 + p_3 + p_4 = 0$ .

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# Constructing Scattering Amplitudes

## Lecture 2: On-shell (BCFW) recursion relations at tree level

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### 1 The BCFW construction

The formulas commonly called “on-shell” or “BCFW” [1] recursion relations are not specific to a particular theory. Their common element is a *construction* based on a *linear momentum shift* in complexified momentum space. In any theory where a valid construction like this can be found, recursion relations are generated. The construction was originally performed and proved in pure Yang-Mills theory, where these kinds of relations among amplitudes had already been guessed (based on properties of loop amplitudes). What is remarkable about the BCFW construction is not just the ease with which it can be varied and applied elsewhere, but also the fact that it tends to produce amplitudes in their most compact, elegant forms.

The idea is to consider the amplitude  $A$  as a (complex) function of external momenta,

$$A = A(p_1, \dots, p_{k-1}, p_k, p_{k+1}, \dots, p_{n-1}, p_n). \quad (1)$$

We introduce a shift of these momenta, *preserving* on-shell conditions and momentum conservation, that is linear in a complex variable  $z$ , and observe the analytic properties of the function  $A(z)/z$ . The only singularities of a Feynman diagram come from its propagators, which, at tree level, lead to simple poles. There is another pole at  $z = 0$ . Now, if  $A(z)$  *vanishes in the limit*  $z \rightarrow \infty$ , then the sum of all residues is zero,

$$0 = \frac{1}{2\pi i} \oint dz \frac{A(z)}{z} \quad (2)$$

$$= A(0) + \sum_{\text{poles } z_\pi \text{ of } A(z)} \text{Res} \left( \frac{A(z)}{z} \right)_{z=z_\pi}. \quad (3)$$

In this equation,  $A(0)$  will be the original, unshifted amplitude, and the other residues will turn out to be factorization limits of the amplitude, in which it

becomes the product of smaller amplitudes. This is because the poles are where propagators go to zero, meaning that there are on-shell amplitudes on either side.

In sum, the two necessary ingredients for this construction to work are a linear momentum shift preserving on-shell and conservation conditions, and the vanishing of  $A(z)$  as  $z \rightarrow \infty$ . The condition of vanishing at infinity is where all the details of the theory come in. Let us first see how to define the momentum shift quite generally, before specializing to massless QCD and using it in detail.

Start by choosing a particle  $j$  with 4-momentum  $p_j$  to shift by a vector  $q$  multiplied by our complex variable  $z$ . It is convenient to denote the shifted momentum by a hat:

$$\widehat{p}_j \equiv p_j(z) \equiv p_j + zq. \quad (4)$$

To conserve momentum overall, this additional term must be absorbed by the momenta of other particles, and the simplest choice is to move it all to one other particle, which we label by  $k$ . Then

$$\widehat{p}_k \equiv p_k(z) = p_k - zq. \quad (5)$$

Along with momentum conservation, we also need the on-shell conditions,

$$\widehat{p}_j^2 = p_j^2 = m_j^2, \quad \widehat{p}_k^2 = p_k^2 = m_k^2, \quad (6)$$

which restrict the choice of  $q$ . It is sufficient to choose  $q$  such that

$$q^2 = 0, \quad q \cdot p_j = 0, \quad q \cdot p_k = 0, \quad (7)$$

and in fact, there are two solutions to these equations (up to rescaling)—provided we can accept *complex* values for momentum.

Whether either of these solutions satisfies  $\lim_{z \rightarrow \infty} A(z) = 0$  depends on the theory. In pure Yang-Mills theory, there is always at least one such  $q$  for any choice of two shifted particles, and for many amplitudes in other theories, there is always at least one choice of shifted particles that gives a valid  $q$ .

For the moment, let us assume that the vanishing condition is satisfied. Then, we need to identify the poles and residues. The poles are located where propagators go to zero. Momentum flowing through a propagator depends on  $z$  if and only if the shifted particles  $j$  and  $k$  are on opposite sides. See Figure 1. For each partition  $\pi$  of the legs into sets separating  $j$  and  $k$ , we use  $P_R$  to denote the momentum flowing through the propagator in the direction of  $k$ , and  $M$  for the mass of the propagating particle. The pole associated to this propagator is the solution  $z = z_\pi$  to the equation  $P_R(z)^2 - M^2 = 0$ . Since  $P_R(z) = P_R - zq$ , the pole is

$$z_\pi = \frac{P_R^2 - M^2}{2q \cdot P_R}, \quad (8)$$

and the residue at this pole is the factorization of the amplitude where the propagator goes on shell,

$$\text{Res} \left( \frac{A(z)}{z} \right)_{z=z_\pi} = A_L(z_\pi) \frac{-i}{P_R^2 - M^2} A_R(z_\pi). \quad (9)$$

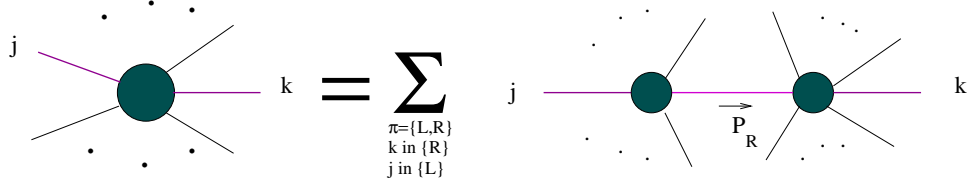


Figure 1: The BCFW recursion relation. The filled circles represent sums of all Feynman diagrams with fixed external legs, which become on-shell amplitudes where the propagator has a pole. The sum is over factorization channels of the amplitude separating the shifted legs  $j$  and  $k$ . The momentum in the propagator is the (shifted) sum of all external momenta on the right-hand side of the diagram.

The propagator should be modified appropriately if it is a fermion. The final recursion relation is therefore

$$A(0) = \sum_{\pi} A_L(z_{\pi}) \frac{i}{P_R^2 - M^2} A_R(z_{\pi}). \quad (10)$$

The sum is over partitions of the external particles, but also over all internal states (helicity, mass, etc.). It is a sum over factorization channels separating  $j$  and  $k$ .

## 2 The 3-point amplitude

The BCFW recursion relation constructs an amplitude from amplitudes with smaller numbers of legs, all the way down to 3. Before we examine the recursion in detail in Yang-Mills theory, it is important to understand the basic building blocks, the 3-point amplitudes. Let us consider the 3-point amplitude of gluons in detail.

Start from the momentum conservation condition. Label the three outward-directed momenta by  $p, q, r$ , which satisfy momentum conservation,  $p+q+r=0$ . Then, since  $r^2=0$ , we have

$$\langle \lambda_p \lambda_q \rangle [\tilde{\lambda}_q \tilde{\lambda}_p] = 0, \quad (11)$$

so that

$$\langle \lambda_p \lambda_q \rangle = 0 \quad \text{or} \quad [\tilde{\lambda}_p \tilde{\lambda}_q] = 0. \quad (12)$$

In *real* Minkowski space, the spinor products are complex conjugates (up to a sign), so they must *both vanish* separately. There are no finite momentum invariants or spinor products, and in fact the amplitude truly vanishes in physical phase space.



However, in *complexified* momentum space (or alternatively, in real momentum space with signature  $++--$ ), the spinors of positive and negative chirality or completely independent, so we can choose just one of the conditions  $\langle \lambda_p \lambda_q \rangle = 0$  or  $[\tilde{\lambda}_q \tilde{\lambda}_p] = 0$  with which to satisfy momentum conservation. These conditions can be rephrased in terms of the proportionality of the 2-component spinors, as  $\lambda_p \sim \lambda_q$  or  $\tilde{\lambda}_p \sim \tilde{\lambda}_q$ . We can repeat this argument starting from  $p^2 = 0$  or  $q^2 = 0$  instead of  $r^2 = 0$ , leading to the conclusions that either all the  $\lambda$ 's are proportional, or else all the  $\tilde{\lambda}$ 's are proportional. In each of these two cases, we can write a nonvanishing expression for a 3-point helicity amplitude. In summary, the three-point gluon amplitudes are given by

$$A(p^+, q^+, r^-) = i \frac{[\tilde{\lambda}_p \tilde{\lambda}_q]^3}{[\tilde{\lambda}_q \tilde{\lambda}_r][\tilde{\lambda}_r \tilde{\lambda}_p]}, \quad \text{and} \quad \langle \lambda_p \lambda_q \rangle = \langle \lambda_p \lambda_r \rangle = \langle \lambda_q \lambda_r \rangle = 0;$$

$$A(p^-, q^-, r^+) = i \frac{\langle \lambda_p \lambda_q \rangle^3}{\langle \lambda_q \lambda_r \rangle \langle \lambda_r \lambda_p \rangle}, \quad \text{and} \quad [\lambda_p \lambda_q] = [\lambda_p \lambda_r] = [\lambda_q \lambda_r] = 0.$$

These formulas can be verified by explicit construction from the Feynman rules with the appropriate polarization vectors. They will be needed in BCFW recursion, where the original unshifted momenta are real-valued, but the shift introduces complex momenta.

### 3 The vanishing condition and recursion relations in QCD

Let us look more specifically at QCD, starting with all-gluon amplitudes. They are color-ordered, so we label the gluons cyclically from 1 to  $n$ . It is obvious that the number of terms in the recursion relation depends on the number of partitions separating the two shifted particles, which is smallest if they are cyclically adjacent. Let us therefore choose adjacent particles to shift and choose the labeling so that they are called 1 and  $n$ , with the shift

$$p_1(z) = p_1 - zq, \quad p_n(z) = p_n + zq. \quad (13)$$

The two solutions of (7) are  $q = \lambda_1 \tilde{\lambda}_n$  and  $q = \lambda_n \tilde{\lambda}_1$ , up to rescaling, of course, and by now it is clear that any scalar factors are taken care of by  $z$ .

We want a solution such that  $\lim_{z \rightarrow \infty} A(z) = 0$ . It turns out that if we choose

$$q = \lambda_1 \tilde{\lambda}_n, \quad (14)$$

then the vanishing condition is satisfied for the helicity choices  $(h_1, h_n) = (+, +)$  or  $(-, -)$  or  $(-, +)$ . By label permutation, then, it is clear that any pair of gluons can be shifted.

We can prove the vanishing condition for  $(h_1, h_n) = (-, +)$  at the level of Feynman diagrams. (For the like-helicity shifts, the vanishing does not hold at

the diagram level, so different arguments are needed, which were given in the BCFW paper.) Notice that the shift can be written in terms of spinors as

$$\lambda_1(z) = \lambda_1, \quad \tilde{\lambda}_1(z) = \tilde{\lambda}_1 - z\tilde{\lambda}_n, \quad (15)$$

$$\lambda_n(z) = \lambda_n + z\lambda_1, \quad \tilde{\lambda}_n(z) = \tilde{\lambda}_n. \quad (16)$$

Now we can just follow the  $z$ -dependence in any possible Feynman diagram. First, notice that the only internal momenta carrying  $z$ -dependence are those along the path connecting gluons 1 and  $n$ . The only way that  $z$  appears in the numerator is from cubic vertices along this path, each linear in  $z$ . If the path contains  $r$  vertices, then there are also  $r - 1$  propagators, each falling off as  $1/z$ . Finally, there are the polarization vectors, and for this helicity choice we can see that both  $\epsilon_1^-$  and  $\epsilon_n^+$  fall off as  $1/z$ . Thus, each diagram goes to zero at infinity, as  $1/z$ .

We finish the construction by identifying the poles and residues. The propagators carrying  $z$ -dependence can be labeled by their momenta,  $P_{1,i}(z)$  (since they separate 1 and  $n$ , we choose to define it in the direction of 1), and the poles labeled simply by  $i$ . If  $z_i$  is the solution to  $P_{1,i}(z)^2 = 0$ , it is given by

$$z_i = \frac{P_{1,i}^2}{\langle 1|P_{1,i}|n \rangle}. \quad (17)$$

The recursion relation is written as follows. There is a sum over internal helicities. Recall that the helicities are restricted to  $(h_1, h_n) = (+, +), (-, -), (-, +)$ . The hats denote shifted momenta, and the shift is different in each term.

$$A_n = \sum_{i=2}^{n-2} \sum_{h=+,-} A(\hat{1}, 2, \dots, i, -\hat{P}_{1,i}^h) \frac{-i}{P_{1,i}^2} A(\hat{P}_{1,i}^{-h}, i+1, \dots, n-1, \hat{n}) \quad (18)$$

$$\hat{p}_1 = \lambda_1 \left( \tilde{\lambda}_1 - \frac{P_{1,i}^2}{\langle 1|P_{1,i}|n \rangle} \tilde{\lambda}_n \right) \quad (19)$$

$$\hat{p}_n = \left( \lambda_n + \frac{P_{1,i}^2}{\langle 1|P_{1,i}|n \rangle} \lambda_1 \right) \tilde{\lambda}_n \quad (20)$$

$$\hat{P}_{1,i} = P_{1,i} - \frac{P_{1,i}^2}{\langle 1|P_{1,i}|n \rangle} \lambda_1 \tilde{\lambda}_n = \frac{(P_{1,i} \cdot \tilde{\lambda}_n) (\lambda_1 \cdot P_{1,i})}{\langle 1|P_{1,i}|n \rangle} \quad (21)$$

**Example:**

Let's compute  $A(1^-, 2^-, 3^+, 4^+, 5^+)$ , using the shift (15), (16). The recursion relation (18) becomes

$$\sum_{h=+,-} \left[ A(\hat{1}, 2, -\hat{P}_{1,2}^h) \frac{-i}{P_{1,2}^2} A(\hat{P}_{1,2}^{-h}, 3, 4, \hat{5}) + A(\hat{1}, 2, 3, -\hat{P}_{1,3}^h) \frac{-i}{P_{1,3}^2} A(\hat{P}_{1,3}^{-h}, 4, \hat{5}) \right].$$

The first term inside the brackets vanishes, because gluons 3, 4, 5 all have positive helicity. Likewise, we are forced to take  $h = +$  in the second term to

get a nonzero result. Using the Parke-Taylor formula for the 3- and 4-point amplitudes, we get

$$A(\widehat{1}^-, 2^-, 3^+, -\widehat{P}_{1,3}^+) \frac{-i}{P_{1,3}^2} A(\widehat{P}_{1,3}^-, 4^+, \widehat{5}^+) = i \frac{\langle \widehat{12} \rangle^3}{\langle 23 \rangle \langle 3\widehat{P}_{1,3} \rangle \langle \widehat{P}_{1,3} 1 \rangle} \frac{1}{P_{1,3}^2} \frac{[45]^3}{[5\widehat{P}_{1,3}] [\widehat{P}_{1,3} 4]}$$

The shifted spinors can be read off directly from equations (19), (20), and (21). In particular, the hatted spinors  $\widehat{1}$  and  $\widehat{5}$  in this expression are exactly the unshifted ones. Using the substitution  $P_{1,3}^2 = P_{4,5}^2 = \langle 45 \rangle [54]$ , it is straightforward to see that we recover the 5-point Parke-Taylor formula. In fact, the Parke-Taylor formula can easily be proved by induction in this way. It is equally simple if the negative-helicity gluons are cyclically non-adjacent.

**Exercise:**

Check the example above. Then compute  $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ .

If we want to shift non-adjacent gluons, the helicity restrictions for the vanishing condition are the same, and the recursion relation looks similar, except that the poles will be indexed by two gluon labels instead of just one.

We can easily include massless fermions and scalars. The same shifts of a pair of gluons will still be valid. Some additional shifts that are equally valid at the diagram level are a gluon-fermion pair with helicities  $(h_g, h_f) = (1, -\frac{1}{2})$  and  $(h_f, h_g) = (\frac{1}{2}, -1)$ . (The asymmetry is traced to the choice of  $q$  in (14).)

Many other shifts have been established in the literature. To give a small start towards further exploration, here are just a few references for QCD [2], amplitudes including massive scalars [3], massive fermions [4, 5], gravitons [6, 7], and strings [8]. In amplitudes with massive particles, the results do tend to rely on finding a pair of massless particles to shift. Lately there have been applications to other kinds of quantities as well, such as integrands for multiloop amplitudes and certain correlation functions.

For some examples of gluon amplitudes derived from this recursion relation, see [9], which predates BCFW. The recursion relations had been guessed, but the proof was not yet known. For example, one can prove the Parke-Taylor formula directly, without having to guess a closed form for the Berends-Giele off-shell current.

## 4 Bonus relations

If the falloff of an amplitude under a BCFW shift is stronger than  $1/z$ , then we can construct even more compact recursion relations, called “bonus” relations. For example, suppose  $A(z) \rightarrow 1/z^2$  as  $z \rightarrow \infty$ . Then instead of applying the residue theorem to  $A(z)/z$  as in (2), we can put an extra linear function in the numerator.

$$0 = \oint \frac{\alpha - z}{\alpha z} A(z). \tag{22}$$

We recover the unshifted amplitude  $A(0)$  as the residue at  $z = 0$ . The recursion relation is

$$A(0) = \sum_{\pi} A_L(z_{\pi}) \frac{\alpha - z_{\pi}}{\alpha} \frac{i}{P_R^2 - M^2} A_R(z_{\pi}). \quad (23)$$

By choosing  $\alpha$  to equal one of the poles  $z_{\pi}$ , we obtain a recursion relation that is one term shorter than usual. If the recursion is applied repeatedly, this gives noticeably more compact results. We now give three examples, also as a way of introducing other theories and relations. They show formulas that were previously known but whose compact form was later traced to bonus relations.

## 4.1 Gravity

In gravity, there is no color decomposition. We have mentioned the KLT (Kawai, Lewellen, Tye) relations between graviton and gauge field amplitudes, derived from a relation between closed and open string amplitudes. From KLT relations, the formula for MHV amplitudes at tree level is [10]

$$A(1^-, 2^-, 3^+, \dots, n^+) = \langle 1\ 2 \rangle^8 \left[ \frac{[1\ 2][n-2, n-1]}{\langle 1, n-1 \rangle N(n)} \left( \prod_{i=1}^{n-3} \prod_{j=i+2}^{n-1} \langle i\ j \rangle \right) \prod_{l=3}^{n-3} (-\langle l|P_{l+1, n-1}|n\rangle) + \mathcal{P}(2, 3, \dots, n-2) \right]$$

where  $N(n) = \prod_{i=1}^{n-1} \prod_{j=i+1}^n \langle i\ j \rangle$  and  $\mathcal{P}(2, 3, \dots, n-2)$  represents the sum over all permutations. Notice that there are  $(n-3)!$  terms.

The BCFW shift is valid for helicities  $(-, +)$ ,  $(+, +)$ ,  $(-, -)$ , just as in Yang-Mills theory, but in gravity the behavior at infinity is  $1/z^2$  for each of these shifts. Straightforward application of the recursion relation gives a formula with  $(n-2)!$  terms, while one of the bonus relations gives the formula above.

Here are two other expressions for the MHV amplitude in gravity [11]. The first has  $(n-2)!$  terms but exhibits nice properties, and the second has  $(n-3)!$  terms and can be traced to a bonus relation. The negative helicities can be on any two of the gravitons.

$$A_{\text{MHV}}^{\text{grav}}(1, 2, \dots, n) = \sum_{\mathcal{P}(3, \dots, n)} F(1, 2, \dots, n), \quad (24)$$

$$A_{\text{MHV}}^{\text{grav}}(1, 2, \dots, n) = \sum_{\mathcal{P}(4, \dots, n)} \frac{\langle 12 \rangle \langle 34 \rangle}{\langle 13 \rangle \langle 24 \rangle} F(1, 2, \dots, n), \quad (25)$$

where

$$F(1, 2, \dots, n) \equiv \langle 1n \rangle [n1] \left( \prod_{s=4}^{n-1} \beta_s \right) A_{\text{MHV}}^{\text{YM}}(1, 2, \dots, n)^2, \quad (26)$$

$$\beta_s \equiv -\frac{\langle s, s+1 \rangle}{\langle 2, s+1 \rangle} \langle 2|P_{3, s-1}|s \rangle. \quad (27)$$

## 4.2 QED

There is no color ordering in QED either. The MHV amplitude for scattering a charged fermion pair into  $n$  photons is very simple [12]:

$$A(q^-, \bar{q}^+; 1^-, 2^+, \dots, n^+) = i \frac{\langle q\bar{q} \rangle^{n-2} \langle 1q \rangle^2}{\prod_{k=2}^n \langle qk \rangle \langle \bar{q}k \rangle}. \quad (28)$$

Although these are MHV amplitudes, the lack of color ordering gives an expression with several terms when BCFW is applied directly. Recently it was found that these amplitudes have a very strong falloff when the charged pair is shifted, namely  $1/z^{n-1}$ . Therefore  $n-2$  linear factors can be inserted into (2), giving a “dressed” recursion relation [13]. For MHV amplitudes, the formula above is recovered, and new compact forms for NMHV (next-to-MHV: 3 negative helicities) and NNMHV amplitudes have been found as well.

## 4.3 Yang-Mills theory

In Yang-Mills theory, the original BCF(W) recursion relation gives the most compact formulas for gluon amplitudes. While shifts of non-adjacent gluons give longer formulas, they have better boundary behavior, so there are bonus relations. In this way new proofs have been found [14] of the familiar  $U(1)$  decoupling identity, a generalization of this identity called the Kleiss-Kuijf relation, and relatively new identities by Bern-Carrasco-Johansson [15]. These last identities were conjectured based on the observation that gauge theory amplitudes can be given in a form where their kinematic factors satisfy identities analogous to the Jacobi identity satisfied by their corresponding color factors. They have been useful for computing multi-loop amplitudes in supergravity. The bonus relations for nonadjacent shifts have also given completely new identities.

All of these identities relate different permutations of labels in  $n$ -point gluon amplitudes. A rephrasing of the  $U(1)$  (or photon) decoupling identity is

$$\sum_{\sigma \in \text{cyclic}} A(1, \sigma(2, 3, \dots, n)) = 0. \quad (29)$$

The Kleiss-Kuijf relation is

$$A(1, \{\alpha\}, n, \{\beta\}) = (-1)^{|\beta|} \sum_{\sigma \in OP(\{\alpha\}, \{\beta^T\})} A(1, \sigma, n), \quad (30)$$

where the sum is over “ordered permutations” preserving the respective orderings of the two subsets, and  $\{\beta^T\}$  is  $\{\beta\}$  with the ordering reversed.

The BCJ relations are more complicated, so we do not write all the details here, but they take the form

$$A(1, 2, \{\alpha\}, 3, \{\beta\}) = \sum_{\sigma \in POP(\{\alpha\}, \{\beta\})} A(1, 2, 3, \sigma) \prod_k^m \mathcal{F}_k, \quad (31)$$



Thus, we have an expansion in terms of MHV amplitudes only. They are built out of spinor products  $\langle a b \rangle$ . Among these, the only appearance of either shifts or propagators is in factors of the form  $\langle a \hat{P}_R \rangle$ . It is easy to replace these. The definition of the shift implies  $\langle a | \hat{P}_R | \tilde{\eta} \rangle = \langle a | P_R | \tilde{\eta} \rangle$ , so

$$\langle a \hat{P}_R \rangle = \frac{\langle a | \hat{P}_R | \tilde{\eta} \rangle}{[\hat{P}_R \tilde{\eta}]} = \frac{\langle a | P_R | \tilde{\eta} \rangle}{[\hat{P}_R \tilde{\eta}]} \quad (35)$$

We know further from the helicity scaling identity that there must be exactly equal numbers of these factors in the numerator and denominator (since the propagator appears once each with positive and negative helicity in  $A_L$  and  $A_R$ ). Therefore, for each of these appearances, we can simply make the substitution

$$\langle a \hat{P}_R \rangle \rightarrow \langle a | P_R | \tilde{\eta} \rangle. \quad (36)$$

These are the ‘‘MHV Rules’’ or ‘‘CSW Rules’’ (Cachazo-Svrček-Witten) for generating tree-level amplitudes. In general, they are given as follows:

- Choose an arbitrary ‘‘reference’’ spinor  $\tilde{\eta}$ .
- Draw all possible graphs with fixed external lines (in cyclic order, if applicable), such that each node has an MHV helicity configuration.
- Propagators are evaluated as usual, with momenta determined by momentum conservation.
- Vertices are the MHV amplitudes! Write them from the Parke-Taylor formula in the case of gluons. Propagators must be continued on-shell: define  $\lambda_P = P \cdot \tilde{\eta}$ .

The definition  $\lambda_P = P \cdot \tilde{\eta}$  is equivalent to the replacement (36). We do not need  $\tilde{\lambda}_P$  because it doesn’t appear in MHV amplitudes.

For a general gluon amplitude with at least 3 negative helicities, the MHV rules can be derived by shifting  $\tilde{\lambda}$  of every negative-helicity gluon. If the negative helicities are on the gluons labeled by  $m_i$ , the shift is

$$\tilde{\lambda}_{m_i}(z) = \tilde{\lambda}_{m_i} + z r_i \tilde{\eta}, \quad (37)$$

where the  $r_i$  are numbers such that  $\sum_i r_i \lambda_{m_i} = 0$ , but no proper subset of these terms sums to zero. (In the NMHV case, the  $r_i$  are determined uniquely.)

Like the BCFW recursion, the MHV rules have been generalized and used in many different contexts. They do not give the most compact formulas, but they are an illuminating expansion, quite direct compared to Feynman diagrams, and with very simple on-shell continuation rules.

## 6 The background field interpretation

There is an interpretation of the BCFW shift and recursion relations in terms of a hard particle moving in a soft background [16]. This interpretation allows us to study the structure and construction of recursion relations without worrying about the exact representations of amplitudes. For example, we can work generally in any dimension  $D \geq 4$ . One can see in general terms what kinds of theories and amplitudes will allow recursion relations. Here, we will show specifically how this formalism can prove the validity of BCFW shifts for gluon amplitudes in the  $(+, +)$  and  $(-, -)$  cases.

Consider the  $z \rightarrow \infty$  limit of the momentum shift (4), (5),

$$p_j(z \rightarrow \infty) \rightarrow zq, \quad p_k(z \rightarrow \infty) \rightarrow -zq. \quad (38)$$

If  $q$  were real, this would be an eikonal limit, but of course  $q$  must be complex. We can still use the idea that these large momenta approximate a single hard particle, and the remaining soft particles can be understood collectively as a classical background. The scattering of the hard particle will then be understood by studying quadratic fluctuations about this background.

Let's revisit pure Yang-Mills theory to see how the formalism works. The gauge field will be expanded as

$$\mathcal{A}_\mu = A_\mu + a_\mu, \quad (39)$$

where  $A_\mu$  is the background field and  $a_\mu$  is the fluctuation. The quadratic Lagrangian is

$$L = -\frac{1}{4} \text{tr} D_{[\mu} a_{\nu]} D^{[\mu} a^{\nu]} + \frac{i}{2} \text{tr}[a_\mu, a_\nu] F^{\mu\nu}. \quad (40)$$

After adding a gauge-fixing term,  $(D_\mu a^\mu)^2$ , we get

$$L = -\frac{1}{4} \text{tr} \eta^{ab} D_\mu a_a D^\mu a_b + \frac{i}{2} \text{tr}[a_a, a_b] F^{ab}. \quad (41)$$

Here we've relabeled the indices in order to display an "enhanced spin symmetry" in the first term: there is something like a Lorentz symmetry acting on the indices of the fluctuation  $a_a$ . (Actual Lorentz invariance is broken by the non-vanishing background field.) Intuitively, this is understood as the conservation of helicity of the hard particle.

In the gauge-fixed Lagrangian, the vertices of  $\mathcal{O}(z)$  come from the first term, with the  $\eta^{ab}$  symmetry. Propagators are still  $\mathcal{O}(1/z)$  along the path of the hard particle (now the only scattered particle).

The full amplitude is

$$A = (\epsilon_j)_a M^{ab} (\epsilon_k)_b, \quad (42)$$

where  $M^{ab}$  is the expression assembled from vertices and propagators, before contracting with the polarization vectors. From the gauge-fixed Lagrangian, we



see that its expansion in  $z$  must take the form

$$M^{ab} = (cz + \dots)\eta^{ab} + A^{ab}\frac{1}{z}B^{ab} + \dots, \quad (43)$$

where  $A^{ab}$  is antisymmetric, because it comes from terms with exactly one insertion of the second-term vertex; and  $B^{ab}$  has no particular symmetry properties; and dots represent terms that are lower order in  $z$ .

What are the polarization vectors? Notice that the polarization vectors satisfy the equations (7) for the shift vector  $q$ . Concretely, we consider one of the two solutions (corresponding to  $q = \lambda_j \tilde{\lambda}_k$  in four dimensions), so that

$$q = \epsilon_j^- = \epsilon_k^+, \quad q^* = \epsilon_j^+ = \epsilon_k^-. \quad (44)$$

These choices can be justified in  $D$  dimensions by looking in the center-of-mass Lorentz frame. The shifted polarization vectors are then

$$\epsilon_j^+(z) = q^* - zp_k, \quad \epsilon_k^-(z) = q^* + zp_j, \quad (45)$$

while  $\epsilon_j^-(z)$  and  $\epsilon_k^+(z)$  are still equal to  $q$ .

We will also use the Ward identities, such as  $(p_j)_a(z)M^{ab}(\epsilon_k)_b = 0$ , which implies

$$q_a M^{ab}(\epsilon_k)_b = -\frac{1}{z}(p_j)_a M^{ab}(\epsilon_k)_b. \quad (46)$$

First, let us review the shift whose vanishing we proved by Feynman diagrams, namely  $M^{-+} = (\epsilon_j^-(z))_a M^{ab}(\epsilon_k^+(z))_b$ . In the background field expansion, it follows from the expansion (43) together with the Ward identity (46) and the fact that  $p_j \cdot q = 0$ , that  $M^{-+} = \mathcal{O}(1/z)$ .

The analysis for  $M^{-+}$  is only slightly more complicated. Here we have

$$M^{--} = (\epsilon_j^-(z))_a M^{ab}(\epsilon_k^-(z))_b \quad (47)$$

$$= q_a M^{ab}(\epsilon_k^-(z))_b \quad (48)$$

$$= -\frac{1}{z}(p_j)_a M^{ab}(\epsilon_k^-(z))_b \quad (49)$$

$$= -\frac{1}{z}(p_j)_a \left[ (cz + \dots)\eta^{ab} + A^{ab} + \frac{1}{z}B^{ab} + \dots \right] (q^* + zp_j)_b \quad (50)$$

$$= \mathcal{O}(1/z). \quad (51)$$

The higher-order terms in (50) drop out because  $p_j^2 = 0$ ,  $p_j \cdot q^* = 0$ , and  $(p_j)_a A^{ab}(p_j)_b = 0$  due to the antisymmetry of  $A^{ab}$ .

In the expansion of  $M^{-+}$ , the leading term is  $-cz^3 p_j \cdot p_k$ , so indeed the boundary behavior does not give a recursion relation.

Additional simplifications are available in the lightcone gauge  $q \cdot A = 0$ . This gauge eliminates  $\mathcal{O}(z)$  behavior in all vertices except those to which both  $j$  and  $k$  are directly attached. Therefore, if we shift non-adjacent gluons, there are no vertices of  $\mathcal{O}(z)$ , and the same arguments show e.g. that  $M^{-+} = \mathcal{O}(1/z^2)$ , leading to the kind of bonus relations we discussed in the previous section. The lightcone gauge  $q \cdot A = 0$  also helps to understand BCFW recursion at the level of Feynman diagrams.

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# Constructing Scattering Amplitudes

## Lecture 3: Twistors and $\mathcal{N} = 4$ Supersymmetric Yang-Mills

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The main theme of these lectures is the construction of scattering amplitudes from their *singularities*. We have seen how complex poles are used in the BCFW construction, and later on we will discover the use of branch cuts and related singularities for loop amplitudes. In this lecture, we take a step aside and consider the construction of amplitudes from their *symmetries*, incarnating the simplicity promised at the start.

First, there is supersymmetry. Color-ordered tree-level QCD is “effectively” supersymmetric. Since there are no loops, there are no superpartner contributions. Moreover, after the color decomposition, quarks can be treated exactly like gluinos. Later, we will see how a “supersymmetry decomposition” is also useful for computing one-loop amplitudes in QCD.  $\mathcal{N} = 1$  supersymmetry is enough to derive useful relations, but we will go directly to  $\mathcal{N} = 4$  supersymmetry here.

$\mathcal{N} = 4$  supersymmetric Yang-Mills (SYM) theory is also conformal. The natural setting for (super)conformal symmetry is *twistor* space, where the symmetry generators are all first-order differential operators. In twistor space, amplitudes are localized on curves, and the components of the curves can be reinterpreted as complete MHV subamplitudes, leading to the “MHV Diagram” method, also called “CSW rules.” These constructions have been generalized to nonsupersymmetric theories with a variety of additional fields.

We will also discuss *dual* superconformal symmetry and the way it enlarges the superconformal symmetry to the full Yangian algebra, an exact symmetry of tree level amplitudes in  $\mathcal{N} = 4$  SYM. The dual space where this symmetry operates is the natural home of Wilson loops whose relationship to scattering amplitudes has been an especially fruitful research topic in recent years. Dual space has its own twistor space, parametrized by “momentum twistor” coordinates, which is perhaps the most natural setting of all for the study of  $\mathcal{N} = 4$  SYM amplitudes.

In this realm, it is conventional to omit the factor of  $i$  in vertices and amplitudes and  $-i$  in propagators, and I now do so as well. Throughout this lecture

we will assume color-ordered amplitudes of massless particles.

**Recommended reading:** In the presentation of twistor space and the localization of amplitudes, I am following Witten [1]. A much fuller introduction to twistor space may be found there, as well as original references, notably to work of Penrose and Nair. I am also drawing heavily on the recent lectures of Drummond [2], which give an introduction to  $\mathcal{N} = 4$  superamplitudes, dual space and the Yangian algebra and all the relevant original references. Material on momentum twistors and the Grassmannian integral follows [3].

## 1 Twistor space

Twistor space can be considered as the space of light rays; it is particularly well suited for studying massless particles. Given the spinor representation of a null momentum 4-vector,  $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$ , the twistor transform replaces  $\tilde{\lambda}_{\dot{a}}$  by another two-component object,  $\mu^{\dot{a}}$ , as follows:

$$\mu^{\dot{a}} = -i \frac{\partial}{\partial \tilde{\lambda}_{\dot{a}}}, \quad \tilde{\lambda}_{\dot{a}} = i \frac{\partial}{\partial \mu^{\dot{a}}}. \quad (1)$$

This can also be called the “half-Fourier” or “Penrose” or “Nair” transform. Notice that it breaks the original parity symmetry between  $\lambda$  and  $\tilde{\lambda}$ . In twistor space, there is no obvious relationship between MHV amplitudes (two negative helicities,  $n - 2$  positive helicities) and the conjugate  $\overline{\text{MHV}}$  amplitudes (with two positive helicities,  $n - 2$  negative helicities).

One motivation for studying Yang-Mills theory in twistor space is that the conformal symmetry generators become first order differential operators. In spinor variables, the generators were given by

$$\begin{aligned} p_{a\dot{a}} &= \lambda_a \tilde{\lambda}_{\dot{a}} & j_{ab} &= \frac{i}{2} \left( \lambda_a \frac{\partial}{\partial \lambda^b} + \lambda_b \frac{\partial}{\partial \lambda^a} \right) \\ k_{a\dot{a}} &= \frac{\partial^2}{\partial \lambda^a \partial \tilde{\lambda}^{\dot{a}}} & \tilde{j}_{ab} &= \frac{i}{2} \left( \tilde{\lambda}_{\dot{a}} \frac{\partial}{\partial \tilde{\lambda}^{\dot{b}}} + \tilde{\lambda}_{\dot{b}} \frac{\partial}{\partial \tilde{\lambda}^{\dot{a}}} \right) \\ d &= \frac{i}{2} \left( \lambda_a \frac{\partial}{\partial \lambda^a} + \tilde{\lambda}_{\dot{a}} \frac{\partial}{\partial \tilde{\lambda}^{\dot{a}}} + 2 \right) \end{aligned}$$

For the symmetries of an amplitude, there is an implicit sum over the particle labels  $i$ . In twistor variables, the generators are

$$\begin{aligned} p_{a\dot{a}} &= i \lambda_a \frac{\partial}{\partial \mu^{\dot{a}}} & j_{ab} &= \frac{i}{2} \left( \lambda_a \frac{\partial}{\partial \lambda^b} + \lambda_b \frac{\partial}{\partial \lambda^a} \right) \\ k_{a\dot{a}} &= i \mu_{\dot{a}} \frac{\partial}{\partial \lambda^a} & \tilde{j}_{ab} &= \frac{i}{2} \left( \mu_{\dot{a}} \frac{\partial}{\partial \mu^{\dot{b}}} + \mu_{\dot{b}} \frac{\partial}{\partial \mu^{\dot{a}}} \right) \\ d &= \frac{i}{2} \left( \lambda_a \frac{\partial}{\partial \lambda^a} - \mu_{\dot{a}} \frac{\partial}{\partial \mu^{\dot{a}}} \right) \end{aligned}$$

Not only are the generators first-order, but the dilatation operator has become homogeneous. The conformal symmetry generators can now be viewed as generating the natural action of  $SL(4)$  on twistor space  $\mathbb{T}$ , which is a copy of  $\mathbb{C}^4$ .

In twistor space, the scaling relation for amplitudes becomes

$$\left( \lambda_i^a \frac{\partial}{\partial \lambda_i^a} - \mu_i^{\dot{a}} \frac{\partial}{\partial \mu_i^{\dot{a}}} \right) A(\lambda_i, \mu_i, h_i) = (-2h_i - 2)A(\lambda_i, \mu_i, h_i), \quad (2)$$

for each of the external particles, indexed by  $i$ . Thus, the amplitude is homogeneous in the twistor coordinates, of fixed degree. It can be viewed as a section of the line bundle  $\mathcal{O}(-2h_i - 2)$  over *projective* twistor space  $\mathbb{P}\mathbb{T}$ . For the study of scattering amplitudes, then, we should treat the twistor coordinates  $(\lambda, \mu)$  as homogeneous coordinates. Projective twistor space  $\mathbb{P}\mathbb{T}$  is three-dimensional<sup>1</sup> and is isomorphic to  $\mathbb{C}\mathbb{P}^3$ .

The external wavefunctions are the twistor transforms of plane waves, which are delta functions:<sup>2</sup>

$$e^{iy \cdot p} \rightarrow \int \frac{d^2 \tilde{\lambda}}{(2\pi)^2} \exp(iy^{a\dot{a}} \cdot \lambda_a \tilde{\lambda}_{\dot{a}}) \exp(i\tilde{\lambda}_{\dot{a}} \mu^{\dot{a}}) = \delta^{(2)}(\mu_{\dot{a}} + y_{a\dot{a}} \lambda^a). \quad (3)$$

The wavefunction is then localized on a space where

$$\mu_{\dot{a}} + y_{a\dot{a}} \lambda^a = 0. \quad (4)$$

This equation is called the “twistor equation” or the “incidence relation.” From it, we can read the geometric correspondence between spacetime and twistor space.

A *point in spacetime*, specified by a fixed  $y$ , defines a *line in twistor space*, on which the two components of  $\lambda^a$  are homogeneous coordinates. In complexified twistor space, this line is a  $\mathbb{C}\mathbb{P}^1$ , isomorphic to a 2-sphere.

A *point in twistor space*, specified by fixed coordinates  $\lambda, \mu$ , defines a so-called  $\alpha$ -plane in spacetime, a two-dimensional subspace whose tangent vectors are all null. If  $y$  and  $y'$  lie on the same  $\alpha$ -plane, then  $\mu_{\dot{a}} + y_{a\dot{a}} \lambda^a = 0$  and  $\mu_{\dot{a}} + y'_{a\dot{a}} \lambda^a = 0$ . It follows that  $(y - y')_{a\dot{a}} \lambda^a = 0$ , meaning that  $\det(y - y')_{a\dot{a}} = 0$ ; in other words,  $y$  and  $y'$  are null separated:  $(y - y')^2 = 0$ .

Two lines in twistor space intersect (at a point) iff the corresponding points in spacetime are null-separated. The intersection point in twistor space corresponds to an  $\alpha$ -plane containing the two spacetime points.

A set of  $n$  points in twistor space are collinear iff the corresponding  $\alpha$ -planes in spacetime intersect at a point. This property will motivate the MHV diagram prescription, which we presented in the previous lecture and will see again shortly.

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<sup>1</sup>Here, complex dimensions. There is also a real version, with all coordinates taking real values. Twistors are most rigorously defined with real coordinates in  $++--$  signature, where for example  $\lambda$  and  $\tilde{\lambda}$  are real-valued and completely independent.

<sup>2</sup>I am using  $y$  as a spacetime coordinate to avoid confusion with  $x$ , which will be a coordinate in dual space later on.

## Localization of amplitudes in twistor space

Witten [1] conjectured that an amplitude in twistor space is localized on an algebraic curve of degree

$$d = \# \text{ negative helicities} + \# \text{ loops} - 1 \quad (5)$$

and genus

$$g \leq \# \text{ loops}. \quad (6)$$

For example, an MHV tree-level amplitude should be localized on a curve of degree 1 and genus 0, i.e. a line. We can see why this is true. The MHV amplitude is expressed by the Parke-Taylor formula. In fact, we do not even need to know the details of that formula. The only relevant fact is that it depends on the  $\lambda_i$  and is independent of the  $\tilde{\lambda}_i$ . However, it is important to replace the momentum-conserving delta function in the amplitude. Thus we write

$$\begin{aligned} A_{\text{tree}}^{\text{MHV}} &= (2\pi)^4 \delta^{(4)} \left( \sum_i \lambda_i^a \tilde{\lambda}_i^{\dot{a}} \right) f(\{\lambda_i\}) \\ &= \int d^4 y \exp \left( i y_{a\dot{a}} \sum_i \lambda_i^a \tilde{\lambda}_i^{\dot{a}} \right) f(\{\lambda_i\}). \end{aligned}$$

In the second line we have made the Fourier transform to spacetime. Now we apply the twistor transform for each of the particles, finding

$$\begin{aligned} &\int d^4 y \int \frac{d^2 \tilde{\lambda}^1}{(2\pi)^2} \cdots \frac{d^2 \tilde{\lambda}^n}{(2\pi)^2} \exp \left( i \sum_i \mu_{i\dot{a}} \tilde{\lambda}_i^{\dot{a}} \right) \exp \left( i y_{a\dot{a}} \sum_i \lambda_i^a \tilde{\lambda}_i^{\dot{a}} \right) f(\{\lambda_i\}) \\ &= \int d^4 y \prod_i \delta^{(2)}(\mu_{i\dot{a}} + y_{a\dot{a}} \lambda_i^a) f(\{\lambda_i\}) \end{aligned}$$

We see that the amplitude has its support where all the twistor coordinates  $(\lambda_i, \mu_i)$  lie on a common line. That is the condition for the delta functions to have overlapping support, at a common point  $y$ .

Since MHV amplitudes are localized on lines in twistor space, which in turn correspond to points in spacetime, one can interpret the MHV amplitudes themselves as local interactions, leading to the MHV diagram prescription described in the previous lecture. See Figure 1.

A consequence in twistor space is that amplitudes are localized on curves that are even more special than indicated in the conjecture (5). The curves of degree  $d > 1$  are actually degenerate, in the sense that they are unions of curves built from lines. This degeneracy can be seen from the fact that they are annihilated by a product of “collinear operators”, first-order differential operators acting on triples of points that vanish when the points are collinear in twistor space. See Figure 2.

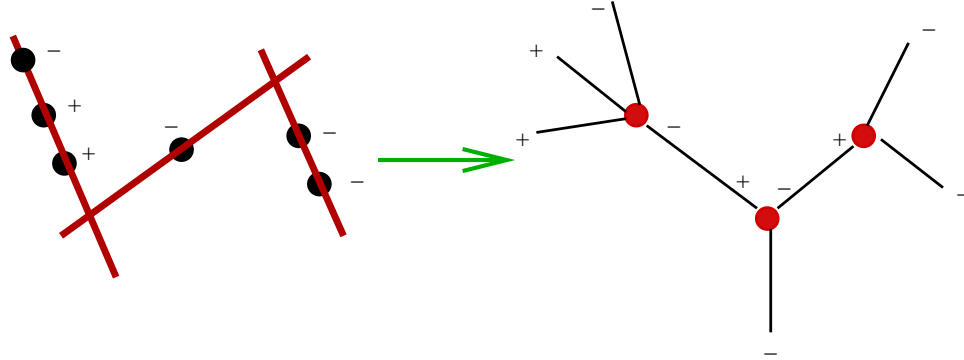


Figure 1: Schematic depiction of MHV Diagrams, in twistor space (left) and spacetime (right). On both sides the amplitude is a sum of such diagrams. On the right side, each vertex is an MHV amplitude whose expression can be read from the Parke-Taylor formula. Each vertex on the right corresponds to a line on the left.

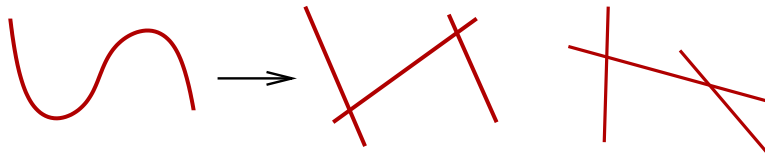


Figure 2: The localization of a tree-level amplitude in twistor space is on a curve of degree  $d$  as given in (5), but which is moreover a union of degenerate curves built from lines.



The MHV diagrams were originally proven to give the correct expressions for amplitudes based on checking that they had all the correct singularities, from collinear limits and multiparticle poles. The constructive proof from a multi-line shift followed later.

## 2 $\mathcal{N} = 4$ SYM

### 2.1 Superfield and superamplitudes

In  $\mathcal{N} = 4$  SYM, gluons on shell live in a PCT self-conjugate supermultiplet with 8 bosonic states and 8 fermionic states. The 16 states can be combined into an on-shell superfield,

$$\Phi = G^+ + \eta^A \Gamma_A + \frac{1}{2!} \eta^A \eta^B S_{AB} + \frac{1}{3!} \eta^A \eta^B \eta^C \epsilon_{ABCD} \bar{\Gamma}^D + \frac{1}{4!} \eta^A \eta^B \eta^C \eta^D \epsilon_{ABCD} G^-. \quad (7)$$

Here, the  $\eta^A$  are Grassmann variables transforming in the fundamental representation of the  $R$ -symmetry algebra  $SU(4)$ , so  $A = 1, 2, 3, 4$ . The on-shell states are  $G^\pm$ , gluons of positive and negative helicity;  $\Gamma_A$  and  $\bar{\Gamma}^A$ , the gluinos and anti-gluinos; and  $S_{AB}$ , 6 real scalars, sometimes combined into 3 complex scalar states. We assign a helicity value of  $\frac{1}{2}$  to  $\eta^A$ , so that the total superfield has helicity 1.

The supersymmetry generators are

$$q^{aA} = \lambda^a \eta^A, \quad \bar{q}_A^{\dot{a}} = \tilde{\lambda}^{\dot{a}} \frac{\partial}{\partial \eta^A}, \quad (8)$$

whose anticommutator is the momentum generator,  $\{q^{aA}, \bar{q}_B^{\dot{a}}\} = p^{a\dot{a}} \delta_B^A$ .

The general form of a superamplitude is

$$A_n = \frac{\delta^{(4)} \left( \sum_i \lambda_i^a \tilde{\lambda}_i^{\dot{a}} \right) \delta^{(8)} \left( \sum_i \lambda_i^a \eta_i^A \right)}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} \mathcal{P}_n(\lambda, \tilde{\lambda}, \eta) \quad (9)$$

The delta function for momentum conservation is joined by its supersymmetric counterpart,  $\delta(\sum_i q_i)$ . The denominator has been included because it shows up naturally in the MHV amplitude, and with these factors built in, the helicity of the remaining function  $\mathcal{P}_n(\lambda, \tilde{\lambda}, \eta)$  is 0.

The other supersymmetry generator,  $\bar{q}_A^{\dot{a}}$ , gives additional conditions on  $\mathcal{P}_n(\lambda, \tilde{\lambda}, \eta)$ . It generates a translation of  $\eta_A$  proportional to  $\tilde{\lambda}^{\dot{a}}$ . Therefore, we must have an invariance of the form

$$\mathcal{P}_n(\eta_i) = \mathcal{P}_n(\eta_i + [\tilde{\lambda}_i \tilde{\zeta}]). \quad (10)$$

This symmetry can be used to translate any two  $\eta_j, \eta_k$  to zero, by choosing

$$\tilde{\zeta}_{jk} = \frac{\tilde{\lambda}_j \eta_k - \tilde{\lambda}_k \eta_j}{[\tilde{\lambda}_j \tilde{\lambda}_k]}. \quad (11)$$

Our first example of a superamplitude is the MHV superamplitude, for which  $\mathcal{P}_n(\lambda, \tilde{\lambda}, \eta) = 1$ .

$$A_n^{\text{MHV}} = \frac{\delta^{(4)}\left(\sum_i \lambda_i^a \tilde{\lambda}_i^{\dot{a}}\right) \delta^{(8)}\left(\sum_i \lambda_i^a \eta_i^A\right)}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} \quad (12)$$

How do we recognize this expression as representing an MHV helicity configuration? The superamplitude can be expanded in the Grassmann variables like the superfield. The delta function of a Grassmann variable is the variable itself, so  $\delta^{(8)}\left(\sum_i \lambda_i^a \eta_i^A\right) = \prod_{a,A} \left(\sum_i \lambda_i^a \eta_i^A\right)$ . We read off the component amplitudes multiplying the various Grassmann polynomials. To illustrate just a couple of the terms in the expansion,

$$A_n^{\text{MHV}} = (\eta_1)^4 (\eta_2)^4 A(G_1^-, G_2^-, G_3^+, \dots, G_n^+) + \cdots \\ + (\eta_1)^4 (\eta_2) (\eta_3)^3 A(G_1^-, \Gamma_2, \bar{\Gamma}_3, \dots, G_n^+) + \cdots \quad (13)$$

In the expansion of the Grassmann delta function, the coefficient of  $\eta_1^1 \eta_2^1$  is  $\langle 12 \rangle$ . The coefficient of  $(\eta_1)^4 (\eta_2)^4$  is  $\langle 12 \rangle^4$ , giving the numerator of the Parke-Taylor formula needed for  $A(G_1^-, G_2^-, G_3^+, \dots, G_n^+)$ . The coefficient of  $(\eta_1)^4 (\eta_2) (\eta_3)^3$  is  $\langle 12 \rangle \langle 13 \rangle^3$ .

Notice that we can call this amplitude MHV but have no need to specify *which* of the particles have opposite helicity. The superamplitude includes *all* possible MHV amplitudes of gluons, simply by taking the  $(\eta_j)^4 (\eta_k)^4$  component to see the amplitude where  $j$  and  $k$  have negative helicities. It includes also amplitudes with more mixed helicity configurations where there are multiple fermions or scalars involved.

**Supersymmetric Ward identities** (SWI) are the relations among different amplitudes in the supermultiplet. They were originally proposed from  $\mathcal{N} = 2$  supersymmetry considerations, although  $\mathcal{N} = 1$  suffices, with a single supersymmetry generator  $Q$ , and a vacuum with unbroken supersymmetry. One example of the SWI is the one given in [4],

$$A(1_g^-, 2_P^-, 3_P^+, 4_g^+, \dots, n_g^+) = \left(\frac{\langle 12 \rangle}{\langle 13 \rangle}\right)^{2h_P} A(1_g^-, 2_s^-, 3_s^+, 4_g^+, \dots, n_g^+), \quad (14)$$

where the subscripts are  $g$  for a gluon,  $s$  for a scalar, and  $P$  for a particle of helicity  $h_P$ ; the particles not listed explicitly are positive-helicity gluons; and  $h_P$  is 0 for a scalar (trivial relation), 1 for a gluon, and  $\frac{1}{2}$  for a fermion. This relation can be derived from similar arguments to those relating the two component amplitudes in (13).

Another example of SWI is the vanishing of the amplitudes with all or all-but-one helicities alike,  $A(\pm, +, +, \dots, +) = 0$ . In the  $\mathcal{N} = 4$  superfield formalism, this is evident from the fact that the 8-dimensional Grassmann delta function requires exactly eight  $\eta$ 's for each component amplitude, so there cannot be any contributions without at least two negative helicities. (For scalar fields, helicity is zero, but positive and negative still have interpretations as particle or antiparticle.)

Of course, we know that the three-point amplitudes are an exceptional case. The MHV amplitude  $A(-, -, +)$  is consistent with the formula (12). However, the conjugate  $\overline{\text{MHV}}$  amplitude  $A(+, +, -)$  doesn't vanish. As we know, in this amplitude all the  $\lambda_i$ 's are proportional, so it cannot appear in the form (9). In fact, the general form (9) privileged the variables  $\lambda$  over  $\tilde{\lambda}$ , both in the explicit denominator factors and in the choice of  $q$  rather than  $\bar{q}$  to enforce supersymmetry in the delta function. To recover the amplitude  $A(+, +, -)$ , we can take the parity conjugate of  $A(-, -, +)$ .

$$A_3^{\overline{\text{MHV}}} = \frac{\delta^{(4)}(\sum_i p_i) \delta^{(8)}\left(\sum_i \tilde{\lambda}_i^{\dot{a}} \bar{\eta}_i^A\right)}{[12][23][31]} \quad (15)$$

Here  $\bar{\eta} = \partial/\partial\eta$  is the Grassmann variable conjugate to  $\eta$ . It can be used in a conjugate representation of the superfield,

$$\Phi = \frac{1}{4!}(\bar{\eta})^4 G^+ + \frac{1}{3!}\bar{\eta}_A \bar{\eta}_B \bar{\eta}_C \epsilon^{ABCD} \Gamma_D + \frac{1}{2!}\bar{\eta}_A \bar{\eta}_B S^{AB} + \bar{\eta}_A \bar{\Gamma}^A + G^-. \quad (16)$$

If we transform  $A_3^{\overline{\text{MHV}}}$  back to a form with the delta function of  $q$  (and hence  $\eta$ ), we find

$$A_3^{\overline{\text{MHV}}} = \frac{\delta^{(4)}(\sum_i p_i) \delta^{(4)}(\eta_1[23] + \eta_2[31] + \eta_3[12])}{[12][23][31]} \quad (17)$$

The fact that the supersymmetric delta function is only four-dimensional in this space reflects the factorization of  $\delta(\sum_i \bar{q}_i)$  when all the spinors  $\lambda_i$  are proportional, which is already accounted for by  $\delta(\sum_i p_i)$ .

## 2.2 Recursion relations

Equipped with the three-point superamplitudes, we can hope to construct all  $n$ -point tree amplitudes in compact forms by BCFW recursion relations. In fact, this has been accomplished recently [5]. Here, I will present the argument to prove the validity of the BCFW construction. Instead of proving a vanishing condition directly, we rely on the vanishing condition of pure gluon amplitudes.

Suppose we shift the momenta of superfields 1 and  $n$  as

$$\lambda_1(z) = \lambda_1, \quad \tilde{\lambda}_1(z) = \tilde{\lambda}_1 - z\tilde{\lambda}_n, \quad (18)$$

$$\lambda_n(z) = \lambda_n + z\lambda_1, \quad \tilde{\lambda}_n(z) = \tilde{\lambda}_n. \quad (19)$$

To preserve the argument of the supersymmetric delta function, the variable  $\eta_1$  must be shifted as well, by

$$\eta_1(z) = \eta_1 - z\eta_n. \quad (20)$$

Recall that  $\bar{q}$ -supersymmetry could be used to translate any two  $\eta$ 's to zero. We do this to set  $\eta_1$  and  $\eta_n$  to zero, with a spinor  $\tilde{\zeta}_{1n}$  as defined in (11). Notice

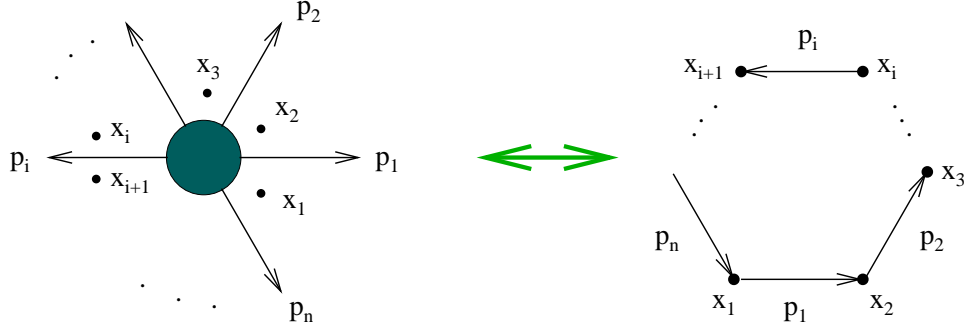


Figure 3: Dual coordinates  $x_j$  label the regions bounded by neighboring momentum vectors  $p_i$  in Minkowski space. In dual space (right), the  $x_j$  are corners of a polygon whose edges are null iff the momenta  $p_i$  are on shell.

that  $\tilde{\zeta}_{1n}$  is invariant under the shift, so we can keep  $\eta_1 = \eta_n = 0$  throughout. That means we are shifting two gluons of positive helicity, which we know is a valid shift (even with the expanded field content). Therefore, this shift is valid for the full superamplitude.

In the recursion relation, the sum over internal helicity states is now replaced by an integral over the internal  $\eta$ .

$$A_n = \sum_{i=2}^{n-1} \int \frac{d^4 \eta_{\hat{P}_i}}{P_i^2} A(\hat{1}, 2, \dots, i, -\hat{P}_{1,i}^h) A(\hat{P}_{1,i}^{-h}, i+1, \dots, n-1, \hat{n}) \quad (21)$$

To write down any superamplitudes beyond the MHV case, it is convenient to use “dual” coordinates, to which we turn next. Because they are associated with other symmetries of  $\mathcal{N} = 4$  SYM, they are also very useful for describing loop amplitudes and Wilson loops.

### 2.3 Dual space

Given a set of momenta  $p_i^{aA}$ , define dual space coordinates  $x_j^{a\hat{a}}$  such that

$$\lambda_i^a \tilde{\lambda}_i^{\hat{a}} = (x_i - x_{i+1})^{a\hat{a}}. \quad (22)$$

Conservation of momentum,  $\sum_i p_i^{aA} = 0$ , then corresponds to the relation

$$x_{n+1} = x_1. \quad (23)$$

These coordinates are also known as ’t Hooft region momenta. They are “dual” in the sense of a planar graph; one can think of the  $x_j$  as labeling the regions bounded by neighboring momentum vectors. (In a planar multiloop amplitude, there would be an additional dual coordinate for each loop.) See Figure 3. Similarly, the supersymmetry generators  $q_i^{aA}$  have dual Grassmann counterparts

$\theta_j^{aA}$  satisfying

$$\lambda_i^a \eta_i^A = \theta_i^{aA} - \theta_{i+1}^{aA}, \quad (24)$$

and likewise  $\sum_i q_i^{aA} = 0$  corresponds to the relation

$$\theta_{n+1} = \theta_1. \quad (25)$$

Within the ‘‘full space’’ parametrized by  $\{\lambda_i, \tilde{\lambda}_i, \eta_i, x_i, \theta_i\}$ , amplitudes are localized on the subspace defined by the constraints (22) and (24). The parametrization by  $\{\lambda_i, \tilde{\lambda}_i, \eta_i\}$  is called on-shell superspace. The parametrization by  $\{\lambda_i, x_i, \theta_i\}$  is called dual chiral superspace. On-shell superspace coordinates can be recovered from dual chiral superspace coordinates by the relations

$$\tilde{\lambda}_i = \frac{(x_i - x_{i+1}) \cdot \lambda_{i+1}}{\lambda_i \cdot \lambda_{i+1}}, \quad \eta_i = \frac{(\theta_i - \theta_{i+1}) \cdot \lambda_{i+1}}{\lambda_i \cdot \lambda_{i+1}}. \quad (26)$$

In dual chiral superspace, the constraints (22) and (24) are rephrased as

$$(x_i - x_{i+1}) \cdot \lambda_i = 0, \quad (\theta_i - \theta_{i+1}) \cdot \lambda_i = 0. \quad (27)$$

The MHV superamplitude is

$$A_n^{\text{MHV}} = \frac{\delta^{(4)}(x_1 - x_{n+1}) \delta^{(8)}(\theta_1 - \theta_{n+1})}{\langle 1 2 \rangle \langle 2 3 \rangle \cdots \langle n 1 \rangle}. \quad (28)$$

In terms of the dual coordinates, we can now write the NMHV superamplitude compactly. First we introduce one more notational convention. Define

$$x_{ij} \equiv x_i - x_j \quad (= p_i + p_{i+1} + \cdots + p_{j-1}), \quad (29)$$

$$\theta_{ij} \equiv \theta_i - \theta_j \quad (= q_i + q_{i+1} + \cdots + q_{j-1}). \quad (30)$$

The NMHV superamplitude is

$$A_n^{\text{NMHV}} = A_n^{\text{MHV}} \mathcal{P}_n^{\text{NMHV}}, \quad \mathcal{P}_n^{\text{NMHV}} = \sum_{r=2}^{n-1} \sum_{s=r+2}^{n-1} R_{n,rs}, \quad (31)$$

where

$$R_{t,rs} = \frac{\langle r, r-1 \rangle \langle s, s-1 \rangle \delta^{(4)}(\langle t | x_{tr} x_{rs} | \theta_{st} \rangle + \langle t | x_{ts} x_{sr} | \theta_{rt} \rangle)}{x_{rs}^2 \langle t | x_{tr} x_{rs} | s \rangle \langle t | x_{tr} x_{rs} | s-1 \rangle \langle t | x_{ts} x_{sr} | r \rangle \langle t | x_{ts} x_{sr} | r-1 \rangle}. \quad (32)$$

## 2.4 Superconformal and dual superconformal symmetry

$\mathcal{N} = 4$  SYM has a superconformal symmetry, whose generators are

$$\begin{aligned} p^{\dot{a}a} &= \sum \tilde{\lambda}^{\dot{a}} \lambda^a, & m_{ab} &= \sum \lambda_{(a} \partial_{b)}, & \bar{m}_{\dot{a}\dot{b}} &= \sum \tilde{\lambda}_{(\dot{a}} \partial_{\dot{b})}, \\ k_{a\dot{a}} &= \sum \partial_a \partial_{\dot{a}}, & d &= \sum \frac{1}{2} \lambda^a \partial_a + \frac{1}{2} \tilde{\lambda}^{\dot{a}} \partial_{\dot{a}} + 1, \end{aligned}$$

$$\begin{aligned}
q^{aA} &= \sum \lambda^a \eta^A, & \bar{q}^{\dot{a}}_A &= \sum \tilde{\lambda}^{\dot{a}} \partial_A, \\
s_{aA} &= \sum \partial_a \partial_A, & \bar{s}^{\dot{a}}_A &= \sum \eta^A \partial_{\dot{a}}, \\
r^A_B &= \sum \eta^A \partial_B + \frac{1}{4} \eta^C \partial_C, \\
c &= \sum 1 + \frac{1}{2} \lambda^a \partial_a - \frac{1}{2} \tilde{\lambda}^{\dot{a}} \partial_{\dot{a}} - \frac{1}{2} \eta^A \partial_A.
\end{aligned}$$

The sums are over the particle labels  $i$ , which are to be understood. It is also to be understood that

$$\partial_a = \frac{\partial}{\partial \lambda^a}, \quad \partial_{\dot{a}} = \frac{\partial}{\partial \tilde{\lambda}^{\dot{a}}}, \quad \partial_A = \frac{\partial}{\partial \eta^A}. \quad (33)$$

The additional symmetry known as dual conformal symmetry is motivated by considering conformal inversion of the dual coordinate, which maps  $x^{a\dot{a}}$  to  $-x^{a\dot{a}}/x^2$  and preserves the null-polygonal property depicted in Figure 3. With respect to the dual coordinates, there is another “dual” copy of the superconformal algebra, whose generators are given by

$$\begin{aligned}
P_{a\dot{a}} &= \sum \partial_{a\dot{a}}, & Q_{aA} &= \sum \partial_{aA}, & \bar{Q}^{\dot{a}}_A &= \sum \theta^{aA} \partial_{a\dot{a}} + \eta^A \partial_{\dot{a}} \\
M_{ab} &= \sum x_{(a}^{\dot{a}} \partial_{b)\dot{a}} + \theta^A_{(a} \partial_{b)} + \lambda_{(a} \partial_{b)}, & \bar{M}_{\dot{a}b} &= \sum x_{(\dot{a}}^a \partial_{b)a} + \tilde{\lambda}_{(\dot{a}} \partial_{b)}, \\
K_{a\dot{a}} &= \sum x_a^{\dot{b}} x_{\dot{a}}^b \partial_{b\dot{b}} + x_a^{\dot{b}} \theta_a^B \partial_{bB} + x_a^{\dot{b}} \lambda_a \partial_b + x_{i+1}^{\dot{b}} \tilde{\lambda}_{\dot{a}} \partial_{\dot{b}} + \tilde{\lambda}_{\dot{a}} \theta_{i+1}^B \partial_B, \\
S_a^A &= \sum -\theta_a^B \theta^{bA} \partial_{bB} + x_a^{\dot{b}} \theta^{bA} \partial_{b\dot{b}} + \lambda_a \theta^{bA} \partial_b + x_{i+1}^{\dot{b}} \eta^A \partial_{\dot{b}} - \theta_{i+1}^B \eta^A \partial_B, \\
\bar{S}_{\dot{a}A} &= \sum x_{\dot{a}}^b \partial_{bA} + \tilde{\lambda}_{\dot{a}} \partial_A, \\
D &= \sum -x^{a\dot{a}} \partial_{a\dot{a}} - \frac{1}{2} \theta^{aA} \partial_{aA} - \frac{1}{2} \lambda^a \partial_a - \frac{1}{2} \tilde{\lambda}^{\dot{a}} \partial_{\dot{a}}, \\
R^A_B &= \sum \theta^{aA} \partial_{aB} + \eta^A \partial_B - \frac{1}{4} \delta_B^A \theta^{aC} \partial_{aC} - \frac{1}{4} \delta_B^A \eta^C \partial_C, \\
C &= \sum -\frac{1}{2} \lambda^a \partial_a + \frac{1}{2} \tilde{\lambda}^{\dot{a}} \partial_{\dot{a}} + \frac{1}{2} \eta^A \partial_A, & B &= \sum \frac{1}{2} \theta^{aA} \partial_{aA} + \lambda^a \partial_a - \tilde{\lambda}^{\dot{a}} \partial_{\dot{a}}.
\end{aligned}$$

Again, the sums are over the particle labels  $i$ , which are implicit everywhere except in the terms of  $K$  and  $S$  where a different label is specified. It should be understood that

$$\partial_{a\dot{a}} = \frac{\partial}{\partial x^{a\dot{a}}}, \quad \partial_{aA} = \frac{\partial}{\partial \theta^{aA}}. \quad (34)$$

While tree-level amplitudes are *invariant* under the original, physical superconformal symmetry, they are *covariant* under the dual superconformal symmetry. They are annihilated by most of the generators; the exceptions are

$$\begin{aligned}
DA_n &= nA_n, & K^{a\dot{a}} A_n &= \left( -\sum_i x_i^{a\dot{a}} \right) A_n, \\
CA_n &= nA_n, & S_a^A A_n &= \left( -\sum_i \theta_{ia}^A \right) A_n.
\end{aligned}$$

This covariant behavior can be seen from the  $A_n^{\text{MHV}}$  prefactor in (9). The factor  $\mathcal{P}_n$  is invariant under both symmetry algebras. In particular, the functions  $R_{t,rs}$  defined in (32) for the NMHV amplitude are invariant under both algebras.

To determine the full symmetry algebra of the  $\mathcal{N} = 4$  SYM tree amplitudes, we subtract weight terms for the covariantly acting generators, defining the new generators

$$D' \equiv D - n, \quad K'^{a\dot{a}} \equiv K^{a\dot{a}} + \sum_i x_i^{a\dot{a}}, \quad S'_a{}^A \equiv S_a{}^A + \sum_i \theta_i^{aA}. \quad (35)$$

The superconformal and modified dual superconformal generators are then combined in on-shell superspace. Here, many generators degenerate or overlap:

$$P = 0, \quad Q = 0, \quad \bar{Q} = \bar{s}, \quad M = m, \quad \bar{M} = \bar{m}, \quad \bar{S} = \bar{q}, \quad D' = \partial, \quad R = r, \quad (36)$$

while  $K'$  and  $S'$  are truly independent additions to the original superconformal algebra. The total symmetry algebra is the graded infinite-dimensional *Yangian* algebra  $Y(PSU(2,2|4))$ . Its level-0 subalgebra is the original superconformal algebra  $PSU(2,2|4)$ .

## 2.5 Momentum twistors and Grassmannian integrals

The symmetry exchanging the superconformal and dual superconformal subalgebras is the transformation between dual space and *its* twistor transform, known as momentum twistor space. This is simply another copy of twistor space, whose coordinates are  $(\lambda^a, \mu_{\dot{a}})$ , with the incidence relation<sup>3</sup>

$$\mu_{\dot{a}} + x_{a\dot{a}}\lambda^a = 0. \quad (37)$$

Compared to the incidence relation (4), the variable  $\lambda$  is identical, but now  $x$  is the coordinate on dual space, so  $\mu$  is completely different. Instead of being related to  $\tilde{\lambda}$  by a Fourier transform, the nonlocal relation is

$$\tilde{\lambda}_i = \frac{\langle i+1, i-1 \rangle \mu_i + \langle i, i+1 \rangle \mu_{i-1} + \langle i-1, i \rangle \mu_{i+1}}{\langle i, i+1 \rangle \langle i, i-1 \rangle}. \quad (38)$$

In the superfield formalism, the momentum twistor also includes Grassmann coordinates  $\chi_A \equiv \theta_{aA}\lambda^a$ . Their relation to the on-shell superspace Grassmann variable is

$$\eta_i = -\frac{\langle i+1, i-1 \rangle \chi_i + \langle i, i+1 \rangle \chi_{i-1} + \langle i-1, i \rangle \chi_{i+1}}{\langle i, i+1 \rangle \langle i, i-1 \rangle}. \quad (39)$$

**Exercise:** Derive the relations (38) and (39) from the definitions of momentum twistors and dual coordinates.

<sup>3</sup>Phase conventions vary in the literature.

Momentum twistors might well be the most natural variables for studying planar amplitudes. In momentum space, conservation of momentum has to be imposed as an additional condition. In dual space, conservation of momentum is automatic, but being on-shell is not (it requires the segments joining the coordinates to be null). Both momentum conservation and on-shell-ness are built into momentum twistor space, so that only any  $n$  coordinates need to be specified.

Recall the Yangian invariant for the NMHV amplitudes given in (32). In momentum twistor variables, it would be written as

$$R_{t,rs} = \frac{\delta^{0|4} (\chi_t \epsilon(r-1, r, s-1, s) + 4 \text{ cyclic permutations})}{\epsilon(t, r-1, r, s-1) \epsilon(r-1, r, s-1, s) \epsilon(r, s-1, s, t) \epsilon(s-1, s, t, r-1) \epsilon(s, t, r-1, r)}. \quad (40)$$

In this form, more of the symmetries of this function are transparent, notably its skew symmetry in five arguments. By itself, this function of the five arguments is  $\mathcal{P}_5^{\text{NMHV}}$ . In general, these cofactors of  $A_n^{\text{MHV}}$  in the superamplitude expression can be computed by a contour integral in a Grassmannian manifold  $G(k, n)$ , which is the space of  $k$ -dimensional planes in  $\mathbb{C}^n$  passing through the origin. (For example,  $G(1, n) = \mathbb{C}\mathbb{P}^{n-1}$ . The Grassmannian contour integral is

$$\mathcal{P}_n^{\text{N}^k\text{MHV}} = \frac{1}{(2\pi i)^{k(n-k)}} \oint_{\Gamma \subset G(k, n)} \prod_{r=1}^k \delta^{4|4}(T^r \cdot Z) \frac{D^{k(n-k)} T}{(12 \cdots k)(23 \cdots k+1) \cdots (n1 \cdots k-1)}. \quad (41)$$

Here,  $\text{N}^k\text{MHV}$  indicates a superamplitude whose components include the pure-gluon amplitude with  $k+2$  negative helicities.  $T = \{T_i^r\}$  is a  $k \times n$  matrix with values in  $\mathbb{C}$ . The variable  $Z$  is the momentum twistor coordinate  $(\lambda, \mu, \chi)$ . The factors in the denominator are the  $k \times k$  determinants of consecutive columns of  $T$ .

Reconstructing the tree amplitude requires choosing a suitable contour  $\Gamma$ , which I will not describe here and is not always clear. However, it has been observed that different contours give rise to the different representations of the tree amplitude, for example by choosing different shifts in BCFW recursion.

It has been argued that these Grassmannian contour integrals are the most general Yangian invariant quantities. Therefore they are expected to give (possibly complete!) information about  $\mathcal{N} = 4$  SYM amplitudes, even to all loop orders.

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# Constructing Scattering Amplitudes, Lecture 4: On-shell techniques for one-loop amplitudes

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We would like to construct scattering amplitudes in terms of their singularities. These can be poles, as in the case of tree amplitudes and the BCFW construction. In loop amplitudes, there are branch cuts, as well as other singularities associated with “generalized” cuts, in which different combinations of propagators are put on shell. All of these singularities probe factorization limits of the amplitude: they select kinematics where some propagators are put *on shell*. Thus, the calculation can be packaged in terms of lower-order *amplitudes* instead of the complete sum of Feynman diagrams.

The “unitarity method” started as a framework for one-loop calculations. Rather than the standard expansion in loop Feynman diagrams, the basic reference point is the linear expansion of the amplitude function in a basis of “master integrals” multiplied by coefficients that are rational functions of the kinematic variables. The point is that the most difficult part of the calculation, namely integration over the loop momentum, can be done once and for all, with explicit evaluations of the master integrals. The master integrals contain all the logarithmic functions. What remains is to find their coefficients.

**References:** Citations to the original works presenting integral reduction, the unitarity method, and consequences of supersymmetry may be found in the familiar review article [1]. More recent improvements are reviewed in my own recent article [2]. Among the more recent works, I indicate a few of the most notable developments as they arise in the text.

## 1 Reduction to master integrals

Integral reduction is a procedure for expressing any one-loop Feynman integral as a linear combination of scalar boxes, scalar triangles, scalar bubbles, and scalar tadpoles, with rational coefficients:

$$A^{1\text{-loop}} = \sum_n \sum_{\mathbf{K}=\{K_1, \dots, K_n\}} c_n(\mathbf{K}) I_n(\mathbf{K}) \quad (1)$$

In four dimensions,  $n$  ranges from 1 to 4. In dimensional regularization, the tadpole contributions with  $n = 1$  arise only with internal masses. If we keep higher order contributions in  $\epsilon$ , we find that the pentagons ( $n = 5$ ) are independent as well.

The traditional reduction procedure proceeds as follows. We assume that the integral has been constructed from Feynman diagrams, so that the denominators are propagators of the form  $D_i = (\ell - K_i)^2 - M_i^2$ , along with the propagator defining the loop momentum,  $D_0 = \ell^2 - M_0^2$ , chosen anew for each term at each stage. Note that then  $2\ell \cdot K_i = M_0^2 + K_i^2 - M_i^2 + D_0 - D_i$ . There are three steps.

First, we eliminate tensor structure (i.e. momentum-dependent numerators) in terms with at least five propagators. Any appearance of  $\ell^2$  in the numerator is replaced by  $M_0^2 + D_0$ , and the  $D_0$  term cancels against the denominator. The remaining momentum dependence in the numerator is polynomial in contractions of the form  $\ell \cdot P$ . Among the five propagators, there are four independent momentum vectors  $K_i$  in which to expand any  $P$ . Then we make the replacement  $2\ell \cdot K_i = M_0^2 + K_i^2 - M_i^2 + D_0 - D_i$  and cancel  $D_0$  and  $D_i$  against the denominator. Step by step, the degree of the polynomial is lowered until we have a scalar numerator or at most four propagators in the denominator.

Second, we eliminate remaining tensor structure in the terms with at most four propagators. This is done by using the momenta appearing in the denominators to build a basis of Lorentz-covariant tensors in which to expand the integral. Contracting the tensors with external momenta gives the constraints needed to solve the linear system. It can be particularly efficient to use contractions with complex momenta constructed from spinors associated to different external legs.

Third, we express  $n$ -point scalar integrals with  $n > 4$  in terms of lower-point scalar integrals. If  $n \geq 6$ , then there is a nontrivial solution  $\{\alpha_i\}$  to the five equations  $\sum_{i=1}^n \alpha_i = 0$  and  $\sum_{i=1}^n \alpha_i P_i^\mu = 0$ . With this solution,  $\sum_i \alpha_i D_i = \sum_i \alpha_i (K_i^2 - M_i^2)$ . Divide the integrand by the (momentum-independent) right-hand side of this equation and multiply it by the left-hand side. The factors  $D_i$  will cancel against the denominator and reduce  $n$  by one. The final remaining concern is the scalar pentagon. If we are keeping full  $\epsilon$  dependence in dimensional regularization, it is an independent master integral. If we truncate the integrals at  $\mathcal{O}(\epsilon)$ , then the scalar pentagon can be reduced further to four scalar boxes. The scalar pentagon integral is finite, so we can now treat its loop momentum as four-dimensional. The final reduction involves expanding it in terms of the axial vectors constructed from triples of four independent external momenta.

## 1.1 Master integrals

The one-loop master integrals are depicted in Figure 1. The scalar  $n$ -point

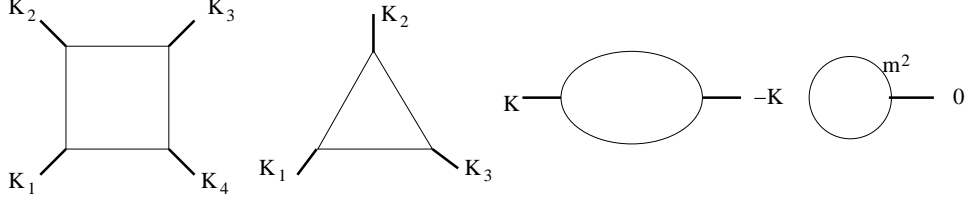


Figure 1: One-loop master integrals: box, triangle, bubble and tadpole. The Lorentz vectors  $K_i$  are sums of external momenta, all directed outward. The tadpole is present if there are internal masses; otherwise, it is zero in dimensional regularization.

integral, without internal masses, is defined as

$$I_n = (-1)^{n+1} i (4\pi)^{\frac{D}{2}} \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{\ell^2 (\ell - K_1)^2 (\ell - K_1 - K_2)^2 \cdots (\ell + K_n)^2}$$

The expressions given below are taken from [3, 4]. All divergent one-loop integrals with possible internal masses may be found in [5]. Other useful expressions for scalar box integrals, convenient for analytic continuation to different kinematic regions, appear in [6].

The dimensional regularization parameter is  $\epsilon = (4 - D)/2$ . The constant  $r_\Gamma$  is defined by

$$r_\Gamma = \frac{\Gamma(1 + \epsilon) \Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \quad (2)$$

**Scalar bubble integral, no internal masses:**

$$I_2 = r_\Gamma \left( \frac{1}{\epsilon} - \ln(-K^2) + 2 \right) + \mathcal{O}(\epsilon) \quad (3)$$

**Scalar triangle integrals, no internal masses:**

If  $K_2^2 = K_3^2 = 0$  and  $K_1^2 \neq 0$ , then the scalar triangle is called “one-mass”, and it is

$$I_3^{1m} = \frac{r_\Gamma}{\epsilon^2} (-K_1^2)^{-1-\epsilon}. \quad (4)$$

If  $K_3^2 = 0$  and  $K_1^2, K_2^2 \neq 0$ , then the scalar triangle is called “two-mass”, and it is

$$I_3^{2m} = \frac{r_\Gamma}{\epsilon^2} \frac{(-K_1^2)^{-\epsilon} - (-K_2^2)^{-\epsilon}}{(-K_1^2) - (-K_2^2)} \quad (5)$$

The “three-mass” scalar triangle is finite and given by

$$I_3^{3m} = \frac{i}{\sqrt{\Delta_3}} \sum_{j=1}^3 \left[ \text{Li}_2 \left( -\frac{1 + i\delta_j}{1 - i\delta_j} \right) - \text{Li}_2 \left( -\frac{1 - i\delta_j}{1 + i\delta_j} \right) \right] + \mathcal{O}(\epsilon), \quad (6)$$

where we have defined the following:

$$\Delta_3 = -(K_1^2)^2 - (K_2^2)^2 - (K_3^2)^2 + 2K_1^2 K_2^2 + 2K_2^2 K_3^2 + 2K_3^2 K_1^2 \quad (7)$$

$$\delta_j = \frac{2K_j^2 - (K_1^2 + K_2^2 + K_3^2)}{\sqrt{\Delta_3}} \quad (8)$$

**Scalar box integrals, no internal masses:**

Let  $s = (K_1 + K_2)^2$  and  $t = (K_1 + K_4)^2$ . The dilogarithm function is defined by  $\text{Li}_2(x) = -\int_0^x \ln(1-z)dz/z$ .

If all four momenta are massless, i.e.  $K_1^2 = K_2^2 = K_3^2 = K_4^2 = 0$  (a special case for four-point amplitudes), then the box integral is given by

$$I_4^{0m} = \frac{2r_\Gamma}{st} \frac{1}{\epsilon^2} [(-s)^{-\epsilon} + (-t)^{-\epsilon}] - \frac{2r_\Gamma}{st} \left[ \frac{1}{2} \ln^2 \left( \frac{s}{t} \right) + \frac{\pi^2}{2} \right] + \mathcal{O}(\epsilon). \quad (9)$$

If only one of the four momenta, say  $K_1$ , is massive, and the other are massless, i.e.  $K_2^2 = K_3^2 = K_4^2 = 0$ , then the box is called ‘‘one-mass’’, and it is given by

$$I_4^{1m} = \frac{2r_\Gamma}{st} \frac{1}{\epsilon^2} [(-s)^{-\epsilon} + (-t)^{-\epsilon} - (-K_1^2)^{-\epsilon}] \quad (10)$$

$$- \frac{2r_\Gamma}{st} \left[ \text{Li}_2 \left( 1 - \frac{K_1^2}{s} \right) + \text{Li}_2 \left( 1 - \frac{K_1^2}{t} \right) + \frac{1}{2} \ln^2 \left( \frac{s}{t} \right) + \frac{\pi^2}{6} \right] + \mathcal{O}(\epsilon).$$

There are two distinct arrangements of two massive and two massless legs on the corners of a box. In the ‘‘two-mass-easy’’ box, the massless legs are diagonally opposite. If  $K_2^2 = K_4^2 = 0$  while the other two legs are massive, the integral is

$$I_4^{2m \ e} = \frac{2r_\Gamma}{st - K_1^2 K_3^2} \frac{1}{\epsilon^2} [(-s)^{-\epsilon} + (-t)^{-\epsilon} - (-K_1^2)^{-\epsilon} - (-K_3^2)^{-\epsilon}] \quad (11)$$

$$- \frac{2r_\Gamma}{st - K_1^2 K_3^2} \left[ \text{Li}_2 \left( 1 - \frac{K_1^2}{s} \right) + \text{Li}_2 \left( 1 - \frac{K_1^2}{t} \right) + \text{Li}_2 \left( 1 - \frac{K_3^2}{s} \right) \right.$$

$$\left. + \text{Li}_2 \left( 1 - \frac{K_3^2}{t} \right) - \text{Li}_2 \left( 1 - \frac{K_1^2 K_3^2}{st} \right) + \frac{1}{2} \ln^2 \left( \frac{s}{t} \right) \right] + \mathcal{O}(\epsilon).$$

In the ‘‘two-mass-hard’’ box, the massless legs are adjacent. If  $K_3^2 = K_4^2 = 0$  while the other two legs are massive, the integral is

$$I_4^{2m \ h} = \frac{2r_\Gamma}{st} \frac{1}{\epsilon^2} \left[ \frac{1}{2} (-s)^{-\epsilon} + (-t)^{-\epsilon} - \frac{1}{2} (-K_1^2)^{-\epsilon} - \frac{1}{2} (-K_2^2)^{-\epsilon} \right] \quad (12)$$

$$- \frac{2r_\Gamma}{st} \left[ -\frac{1}{2} \ln \left( \frac{s}{K_1^2} \right) \ln \left( \frac{s}{K_2^2} \right) + \frac{1}{2} \ln^2 \left( \frac{s}{t} \right) \right.$$

$$\left. + \text{Li}_2 \left( 1 - \frac{K_1^2}{t} \right) + \text{Li}_2 \left( 1 - \frac{K_2^2}{t} \right) \right] + \mathcal{O}(\epsilon).$$

If exactly one leg is massless, say  $K_4^2 = 0$ , then we have the ‘‘three-mass’’ box, given by

$$I_4^{3m} = \frac{2r_\Gamma}{st - K_1^2 K_3^2} \frac{1}{\epsilon^2} \left[ \frac{1}{2} (-s)^{-\epsilon} + \frac{1}{2} (-t)^{-\epsilon} - \frac{1}{2} (-K_1^2)^{-\epsilon} - \frac{1}{2} (-K_3^2)^{-\epsilon} \right] \quad (13)$$

$$-\frac{2r_\Gamma}{st - K_1^2 K_3^2} \left[ -\frac{1}{2} \ln\left(\frac{s}{K_1^2}\right) \ln\left(\frac{s}{K_2^2}\right) - \frac{1}{2} \ln\left(\frac{t}{K_2^2}\right) \ln\left(\frac{t}{K_3^2}\right) \right. \\ \left. + \frac{1}{2} \ln^2\left(\frac{s}{t}\right) + \text{Li}_2\left(1 - \frac{K_1^2}{t}\right) + \text{Li}_2\left(1 - \frac{K_3^2}{s}\right) - \text{Li}_2\left(1 - \frac{K_1^2 K_3^2}{st}\right) \right] + \mathcal{O}(\epsilon).$$

Finally, the ‘‘four-mass’’ box, which is finite, is given by

$$I^{4m} = \frac{1}{a(x_1 - x_2)} \sum_{j=1}^2 (-1)^j \left( -\frac{1}{2} \ln^2(-x_j) \right. \\ - \text{Li}_2\left(1 + \frac{-K_3^2 - i\epsilon}{-s - i\epsilon} x_j\right) - \eta\left(-x_k, \frac{-K_3^2 - i\epsilon}{-s - i\epsilon}\right) \ln\left(1 + \frac{-K_3^2 - i\epsilon}{-s - i\epsilon} x_j\right) \\ - \text{Li}_2\left(1 + \frac{-t - i\epsilon}{-K_1^2 - i\epsilon} x_j\right) - \eta\left(-x_k, \frac{-t - i\epsilon}{-K_1^2 - i\epsilon}\right) \ln\left(1 + \frac{-t - i\epsilon}{-K_1^2 - i\epsilon} x_j\right) \\ \left. + \ln(-x_j)(\ln(-K_1^2 - i\epsilon) + \ln(-s - i\epsilon) - \ln(-K_4^2 - i\epsilon) - \ln(-K_2^2 - i\epsilon)) \right).$$

Here we have defined

$$\eta(x, y) = 2\pi i [\vartheta(-\text{Im } x) \vartheta(-\text{Im } y) \vartheta(\text{Im}(xy)) - \vartheta(\text{Im } x) \vartheta(\text{Im } y) \vartheta(-\text{Im}(xy))],$$

and  $x_1$  and  $x_2$  are the roots of a quadratic polynomial:

$$ax^2 + bx + c + i\epsilon d = a(x - x_1)(x - x_2), \quad (14)$$

with

$$a = tK_3^2, \quad b = st + K_1^2 K_3^2 - K_2^2 K_4^2, \quad c = sK_1^2, \quad d = -K_2^2.$$

## 1.2 Four-dimensional truncation

For practical purposes, we make use of the expansions in  $\epsilon$  given above for the master integrals. Their coefficients also have higher-order dependence on  $\epsilon$ . When these higher-order terms combine with  $1/\epsilon$  ultraviolet divergences (from the scalar bubble), they produce additional rational terms of  $\mathcal{O}(\epsilon^0)$ . The four-dimensional expansion is then

$$A^{1\text{-loop}} = \sum_n \sum_{\mathbf{K}=\{K_1, \dots, K_n\}} c_n(\mathbf{K}) I_n(\mathbf{K}) + \text{rational terms} + \mathcal{O}(\epsilon), \quad (15)$$

where the coefficients  $c_n(\mathbf{K})$  are now independent of  $\epsilon$ .

We work in the four-dimensional-helicity (FDH) scheme of dimensional regularization, in which the polarization vectors remain exactly four-dimensional. Only the loop momentum is continued to  $(4 - 2\epsilon)$  dimensions.

## 2 The unitarity method

The unitarity cut of a one-loop amplitude is its discontinuity across the branch cut in a kinematic region associated to a particular momentum channel. The

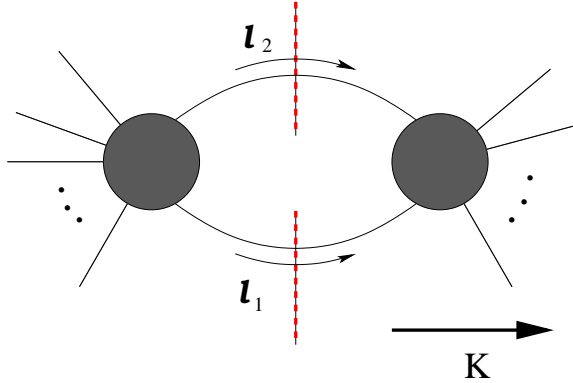


Figure 2: Unitarity cut of a one-loop amplitude in the  $K$  momentum channel. The two propagators are constrained to their respective mass shells. The disks represent the sum of all Feynman diagrams linking the fixed external lines and the two cut propagators.

name comes from the unitarity of the S-matrix: since  $S^\dagger S = 1$ , and we expand  $S = 1 + iT$  where  $T$  is the interaction matrix, then  $2\text{Im } T = T^\dagger T$ . Expanding this equation perturbatively in the coupling constant, we see that the imaginary part of the one-loop amplitude is related to a product of two tree-level amplitudes. Effectively, in the complete sum of Feynman diagrams, two chosen propagators within the loop are restricted to their mass shells. This imaginary part should be viewed more generally as a discontinuity across a branch cut singularity of the amplitude—in a kinematic configuration where one kinematic invariant, say  $K^2$ , is positive, while all others are negative. This condition isolates the momentum channel  $K$  of interest;  $K$  is the sum of some of the external momenta. We will take cuts in various momentum channels to construct the amplitude.

For a one-loop amplitude, the value of the unitarity cut is given by Cutkosky rules, which are expressed in the cut integral,

$$\Delta A^{1\text{-loop}} \equiv \int d\mu \, A_{\text{Left}}^{\text{tree}} \times A_{\text{Right}}^{\text{tree}}, \quad (16)$$

where the Lorentz-invariant phase space (LIPS) measure is defined by

$$d\mu = d^4\ell_1 \, d^4\ell_2 \, \delta^{(4)}(\ell_1 + \ell_2 - K) \, \delta^{(+)}(\ell_1^2) \, \delta^{(+)}(\ell_2^2). \quad (17)$$

Here, the superscript (+) on the delta functions for the cut propagators denotes the choice of a positive-energy solution.

How can these unitarity cuts be used to calculate the amplitude? By applying the cut  $\Delta$  in various momentum channels, we get information about the coefficients of master integrals.

Consider applying a unitarity cut to the expansion (1) or (15) of an amplitude in master integrals. Since the coefficients are rational functions, the branch cuts are located only in the master integrals. Thus we find that

$$\Delta A^{1\text{-loop}} = \sum_n \sum_{\mathbf{K}=\{K_1, \dots, K_n\}} c_n(\mathbf{K}) \Delta I_n(\mathbf{K}). \quad (18)$$

Our task is to isolate the individual coefficients  $c_n(\mathbf{K})$ .

Equation (18) is the key to the unitarity method. It has two important features. First, we see from (16) that it is a relation involving *tree-level* quantities. Second, many of the terms on the right-hand side vanish, because only a subset of master integrals have a cut involving the given momentum  $K$ . Meanwhile, we enjoy the freedom of using all possible values of  $K$  in turn. In effect, we have traded the original single equation (1) for a system of several shorter equations.

One-loop amplitudes are known to be *cut-constructible*. That is, they are uniquely determined by their branch cuts. The cuts should properly be evaluated in  $D$  dimensions. It is usually easier to evaluate four-dimensional cuts, giving the  $\epsilon$ -independent coefficients in (15), and then compute the rational terms separately.

We will now address the problem of finding the coefficients  $c_n(\mathbf{K})$  from four-dimensional cuts.

### 3 Generalized unitarity

Before learning to evaluate unitarity cuts and extract the coefficients of master integrals, let us do something even more direct. Unitarity cuts can be “generalized” in the sense of putting a different number of propagators on shell. This operation selects different kinds of singularities of the amplitude; they are not physical momentum channels like ordinary cuts and do not have an interpretation relating to the unitarity of the S-matrix. Here, it becomes essential to work with complexified momenta.

The most direct application of generalized unitarity is to use a “quadruple cut” to find any box coefficient [7]. If we cut *four* propagators—equivalent to specifying a partition  $(K_1, K_2, K_3, K_4)$  of the external momenta—then the four-dimensional integral becomes trivial. See Figure 3.

$$\Delta_4 A^{1\text{-loop}} = \int d^4\ell \delta(\ell_1^2) \delta(\ell_2^2) \delta(\ell_3^2) \delta(\ell_4^2) A_1^{\text{tree}} A_2^{\text{tree}} A_3^{\text{tree}} A_4^{\text{tree}} \quad (19)$$

Applied to the master integrals, the quadruple cut picks up a contribution from exactly one box integral, namely the one with momenta  $(K_1, K_2, K_3, K_4)$  at the corners. Therefore, the cut expansion collapses to a single term:

$$\Delta_4 A^{1\text{-loop}} = c_4(K_1, K_2, K_3, K_4) \Delta_4 I_4(K_1, K_2, K_3, K_4). \quad (20)$$

The quadruple cut of the scalar box integral is a Jacobian factor which is equal



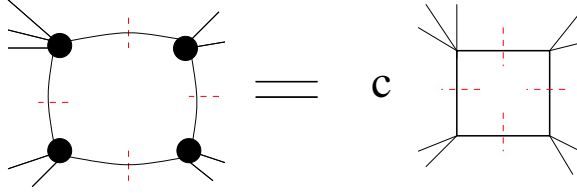


Figure 3: A quadruple cut puts four propagators on shell. It is a trivial integral isolating a single box coefficient.

on both sides of the equation. The result for the coefficient is simply

$$c_4 = \frac{1}{2} \sum_{\ell \in \mathcal{S}} A_1^{\text{tree}}(\ell) A_2^{\text{tree}}(\ell) A_3^{\text{tree}}(\ell) A_4^{\text{tree}}(\ell), \quad (21)$$

where  $\mathcal{S}$  is the solution set for the four delta functions of the cut propagators,

$$\mathcal{S} = \{\ell \mid \ell^2 = 0, \quad (\ell - K_1)^2 = 0, \quad (\ell - K_1 - K_2)^2 = 0, \quad (\ell + K_4)^2 = 0\}. \quad (22)$$

There are exactly two solutions, provided that momenta are allowed to take complex values. This is the origin of the factor of 2 in the denominator of (21). Thus it is easy to get all the box coefficients.

Other applications of “generalized unitarity” include triple cuts for triangle coefficients, single cuts for tadpole and other coefficients, and all possible extensions of these operations in multiloop amplitudes. In particular, the parametrizations of Forde [8] are useful for triple cuts and can be extended to double cuts as well.

## 4 Supersymmetric theories, and SUSY decomposition for QCD

- Both  $\mathcal{N} = 4$  SYM and  $\mathcal{N} = 8$  supergravity have *only* box integrals in the one-loop expansion. This property follows from supersymmetric cancellations combined with power-counting arguments. Consequently, the quadruple cut operation suffices in these cases.
- Supersymmetric massless theories, in general, are *4d cut-constructible*. That is, the four-dimensional cuts determine the one-loop amplitudes completely, and there are no rational terms in (15).
- Supersymmetric one-loop gluon amplitudes are also an ingredient in pure QCD amplitudes. Based on the effective equivalence of gluinos and quarks in color-ordered Feynman rules, and in view of the previous two points, it is useful to rewrite a gluon or quark loop in an amplitude with external gluons as a component of a supersymmetric multiplet. For the gluon, this is the  $\mathcal{N} = 4$  supermultiplet with a gluon  $g$  with four fermions  $f$  and

three complex scalars  $s$ . For a fermion, it is the  $\mathcal{N} = 1$  chiral matter supermultiplet with one fermion and one complex scalar.

$$\begin{aligned} A^{\text{gluon loop}} &= g = (g + 4f + 3s) - 4(f + s) + s = A^{\mathcal{N}=4} - 4A^{\mathcal{N}=1} + A^{\text{scalar}} \\ A^{\text{quark loop}} &= f = (f + s) - s = A^{\mathcal{N}=1} - A^{\text{scalar}} \end{aligned}$$

With this decomposition into  $\mathcal{N} = 4$ ,  $\mathcal{N} = 1$ , and  $\mathcal{N} = 0$  components, the nonzero coefficients are the following:

$\mathcal{N}$	Box	Triangle	Bubble	Rational
4	✓			
1	✓	✓	✓	
0	✓	✓	✓	✓

Because the component with the scalar loop does not propagate spin information, it is simpler than the full gluon loop even though it is still nonsupersymmetric.

## 5 Evaluation of cut integrals and solutions for coefficients

### 5.1 Cuts of master integrals

The utility of equation (16) depends on knowing the master integrals and hence their branch cuts. Having listed the master integrals for massless theories in the previous section, we can calculate the cuts explicitly by taking the imaginary parts of these functions in various kinematic regions. Recall that the kinematic region associated to the unitarity cut in momentum channel  $K$  has  $K^2 > 0$  and all other invariants negative. In the limit of large  $K^2$ , we see that the behavior of the master integrals features uniquely identifiable products of logarithms [3]. It follows that no linear combination of master integrals with rational coefficients, in a given momentum channel, can be cut-free.

The cuts of the bubble integrals are purely rational; this is easily seen from the ordinary logarithm in equation (3). The cuts of all other master integrals are logarithmic. The various arguments of the logarithms identify the original master integrals.

We need to evaluate the left hand side of equation (18), by carrying out the 2-dimensional integral of (16). This is neatly accomplished by the Cauchy residue theorem, in a technique known as “spinor integration” [9]. We now illustrate the technique applied to the master integrals themselves, starting from their definition in (2).

To implement the cut conditions, it is convenient to reparametrize the loop momentum in terms of spinor variables. Now it is crucial that our cut is in 4 dimensions.

Since  $\ell_1$  is null, we can parametrize it with

$$(\ell_1)_{a\dot{a}} = t\lambda_a\tilde{\lambda}_{\dot{a}}, \quad (23)$$

where  $\lambda_a, \tilde{\lambda}_{\dot{a}}$  are *homogeneous* spinors (taking values in  $\mathbb{CP}^1$ ), and  $t$  takes non-negative values. The original loop momentum is real-valued, so we will integrate over the contour where  $\tilde{\lambda}$  is the complex conjugate of  $\lambda$ . In the integral measure, we make the replacement

$$\int d^4\ell_1 \delta^{(+)}(\ell_1^2)(\bullet) = - \int_0^\infty \frac{t}{4} dt \int_{\tilde{\lambda}=\bar{\lambda}} \langle \lambda d\lambda \rangle [\tilde{\lambda} d\tilde{\lambda}] (\bullet). \quad (24)$$

Now we make this substitution explicitly in the second delta function of the LIPS measure defined in (17). The momentum of the second cut propagator is  $(\ell_2)_{a\dot{a}} = K_{a\dot{a}} - t\lambda_a\tilde{\lambda}_{\dot{a}}$ , so the measure becomes

$$\int d\mu (\bullet) = - \int_0^\infty \frac{t}{4} dt \int_{\tilde{\lambda}=\bar{\lambda}} \langle \lambda d\lambda \rangle [\tilde{\lambda} d\tilde{\lambda}] \delta(K^2 - t\langle \lambda|K|\tilde{\lambda} \rangle)(\bullet). \quad (25)$$

This second delta function sets  $t$  to the value

$$t = \frac{K^2}{\langle \lambda|K|\tilde{\lambda} \rangle} \quad (26)$$

so, taking account of the prefactor  $\langle \lambda|K|\tilde{\lambda} \rangle$  of  $t$  inside the delta function, we can now perform the  $t$ -integral trivially:

$$\int d\mu (\bullet) = - \int_{\tilde{\lambda}=\bar{\lambda}} \langle \lambda d\lambda \rangle [\tilde{\lambda} d\tilde{\lambda}] \frac{K^2}{4\langle \lambda|K|\tilde{\lambda} \rangle^2} (\bullet). \quad (27)$$

The remaining integral over the spinor variables is carried out with the residue theorem. We will see how this is done in the master integrals before proceeding to the case of general amplitudes.

### 5.1.1 Cut bubble

Let us start with the scalar bubble integral. The integrand consists entirely of the two cut propagators, so the cut is simply the integral of the LIPS measure,

$$\Delta\left(\frac{1}{\ell^2(\ell-K)^2}\right) = \int d\mu = - \int_{\tilde{\lambda}=\bar{\lambda}} \langle \lambda d\lambda \rangle [\tilde{\lambda} d\tilde{\lambda}] \frac{K^2}{4\langle \lambda|K|\tilde{\lambda} \rangle^2}. \quad (28)$$

In calculating coefficients of complete amplitudes, it can suffice to leave the cut bubble in the form (28) and work at the integrand level. Here we continue and show how to apply the residue theorem to complete the integral. We make use

of the following identity to rewrite the integrand as a total derivative. Here  $\eta$  is an arbitrary spinor.

$$[\tilde{\lambda} d\tilde{\lambda}] \frac{1}{\langle \lambda | K | \tilde{\lambda} \rangle^2} = [d\tilde{\lambda} \partial_{\tilde{\lambda}}] \left( \frac{[\tilde{\lambda} \eta]}{\langle \lambda | K | \eta \rangle \langle \lambda | K | \tilde{\lambda} \rangle} \right). \quad (29)$$

However, the integral is not identically zero, because there are delta-function contributions along the contour. In the theory of a complex variable, we know that

$$\frac{\partial}{\partial \bar{z}} \frac{1}{(z-b)} = 2\pi \delta(z-b). \quad (30)$$

Therefore, we pick up a residue at the pole  $|\lambda\rangle = |K|\eta\rangle$ . Along the contour, since  $\lambda$  and  $\tilde{\lambda}$  are conjugates, we also substitute  $|\tilde{\lambda}\rangle = |K|\eta\rangle$ . The result of the four-dimensional cut bubble is thus

$$\Delta I_2 = \frac{i}{\pi^2} \Delta \left( \frac{1}{\ell^2(\ell-K)^2} \right) = -\frac{iK^2}{2\pi} \left( \frac{[\tilde{\lambda} \eta]}{\langle \lambda | K | \tilde{\lambda} \rangle} \right) \Big|_{|\lambda\rangle=|K|\eta\rangle} = \frac{1}{2\pi i}. \quad (31)$$

(Different conventions in the literature yield results with different powers of  $i$  and  $2\pi$ ; these will be unimportant as long as the framework is consistent. )

### 5.1.2 Cut triangle

In the unitarity cut of the scalar triangle, there is one propagator left over along with the LIPS measure. Converting to the spinor variables, this factor is  $(\ell + K_3)^2 = t \langle \lambda | K_3 | \tilde{\lambda} \rangle + K_3^2$ . Performing the  $t$  integral as before, making the substitution (26) throughout, we have

$$\Delta \left( \frac{1}{\ell^2(\ell-K)^2(\ell+K_3)^2} \right) = - \int_{\tilde{\lambda}=\tilde{\lambda}} \langle \lambda d\lambda \rangle [\tilde{\lambda} d\tilde{\lambda}] \frac{1}{4 \langle \lambda | K | \tilde{\lambda} \rangle \langle \lambda | Q | \tilde{\lambda} \rangle}, \quad (32)$$

where

$$Q = \frac{K_3^2}{K^2} K + K_3. \quad (33)$$

Again, it is worth leaving the expression in the form (32), but let us see how to finish the integral. The two factors in the denominator can be combined with a Feynman parameter, and the spinor integral done just as in the bubble case, so that we have

$$- \int_0^1 dx \int_{\tilde{\lambda}=\tilde{\lambda}} \langle \lambda d\lambda \rangle [\tilde{\lambda} d\tilde{\lambda}] \frac{1}{4 \langle \lambda | (1-x)K + xQ | \tilde{\lambda} \rangle} = \frac{\pi}{2} \int_0^1 dx \frac{1}{((1-x)K + xQ)^2}.$$

The result for the cut in the  $K$ -channel is

$$\begin{aligned}\Delta I_3 &= -\frac{i}{\pi^2} \Delta \left( \frac{1}{\ell^2(\ell - K)^2(\ell - K_3)^2} \right) \\ &= \frac{1}{2\pi i \sqrt{-\Delta_3}} \ln \left( \frac{-2(K_3^2 + K \cdot K_3) + \sqrt{-\Delta_3}}{-2(K_3^2 + K \cdot K_3) - \sqrt{-\Delta_3}} \right),\end{aligned}\quad (34)$$

where  $\Delta_3$  is defined in (7). Notice that this result is logarithmic, as expected. Moreover, it is clear that all three-mass triangles are uniquely identified by the functions  $\Delta_3$ , which play a distinguished role in the expression as the arguments of square roots. (For one-mass and two-mass triangles,  $\Delta_3$  is a perfect square, so the square roots disappear from the formula while the logarithm remains.)

### 5.1.3 Cut box

The calculation of the cut scalar box integral is similar. Now there are two uncut propagators identifying the box, which we write as  $(\ell - K_i)^2$  and  $(\ell - K_j)^2$ . Converting to the spinor variables, they become  $K_i^2 - t \langle \lambda | K_i | \tilde{\lambda} \rangle$  and  $K_j^2 - t \langle \lambda | K_j | \tilde{\lambda} \rangle$ , respectively. Performing the  $t$  integral and making the substitution (26) throughout, we have

$$\Delta \left( \frac{1}{\ell^2(\ell - K)^2(\ell - K_i)^2(\ell - K_j)^2} \right) = - \int_{\tilde{\lambda}=\lambda} \langle \lambda | d\lambda \rangle [\tilde{\lambda} | d\tilde{\lambda}] \frac{1}{4K^2 \langle \lambda | Q_i | \tilde{\lambda} \rangle \langle \lambda | Q_j | \tilde{\lambda} \rangle}, \quad (35)$$

where now we define  $Q_i, Q_j$  by

$$Q_i \equiv \frac{K_i^2}{K^2} K - K_i, \quad Q_j \equiv \frac{K_j^2}{K^2} K - K_j. \quad (36)$$

Here again, we can evaluate the integral by introducing a Feynman parameter. It takes a form similar to the triangle. The final result is

$$\Delta I_4 = \frac{1}{(2\pi i) 2K^2 \sqrt{\Delta_{ij}}} \ln \left( \frac{Q_i \cdot Q_j + \sqrt{\Delta_{ij}}}{Q_i \cdot Q_j - \sqrt{\Delta_{ij}}} \right), \quad (37)$$

where

$$\Delta_{ij} \equiv (Q_i \cdot Q_j)^2 - Q_i^2 Q_j^2. \quad (38)$$

We see that the cut is again logarithmic. One can check that for either of the two choices of the cut configuration (straight across the box or selecting one corner), the function  $\Delta_{ij}$  under the square root corresponds to the discriminant of the quadratic polynomial in (14). In the cases where any of the corners of the box is a null momentum,  $\Delta_{ij}$  is a perfect square.

## 5.2 Unitarity cut of the amplitude

To evaluate the cuts of master integrals, we needed only the relatively simple identity (29), from which we could find the residues according to (30). The cut of the full amplitude will generally have more complicated dependence on the loop momentum. We use partial fraction identities to split long products of denominator factors, effectively doing a reduction of the cut integrand.

Suppose we take the expressions for the tree amplitudes in the cut integral (16) from the Feynman rules. The integrand is a rational function whose denominator is a product of propagator factors of the form  $(\ell - K_i)^2$ . As we have seen the cuts of master integrals, such a factor becomes

$$(\ell - K_i)^2 = \frac{K^2 \langle \lambda | Q_i | \tilde{\lambda} \rangle}{\langle \lambda | K | \tilde{\lambda} \rangle}. \quad (39)$$

Other factors of  $\langle \lambda | K | \tilde{\lambda} \rangle$  arise from the integral measure and the substitution for  $t$  found in (26). The key property is that the denominator of the integrand consists of factors of  $\langle \lambda | Q_i | \tilde{\lambda} \rangle$ , where no two  $Q_i$  are the same, along with some power of the factor  $\langle \lambda | K | \tilde{\lambda} \rangle$ .

We find it helpful to rearrange the integrand in order to identify the cuts of master integrals as given in (28), (32), and (35). This task is accomplished by partial fraction identities that split the denominator factors and reduce the power of  $\langle \lambda | K | \tilde{\lambda} \rangle$  if necessary. In effect, it is a reduction technique for the cut integrals.

The splitting of factors with partial fractions proceeds as follows. First, split the factors  $\langle \ell | Q_j | \ell \rangle$  among themselves, with the following identity:

$$\frac{\prod_{j=1}^{k-1} [a_j \tilde{\lambda}]}{\prod_{i=1}^k \langle \lambda | Q_i | \tilde{\lambda} \rangle} = \sum_{i=1}^k \frac{1}{\langle \lambda | Q_i | \tilde{\lambda} \rangle} \frac{\prod_{j=1}^{k-1} [a_j | Q_i | \tilde{\lambda} \rangle]}{\prod_{m=1, m \neq i}^k \langle \lambda | Q_m | \tilde{\lambda} \rangle}. \quad (40)$$

Next, reduce the power of  $\langle \ell | K | \ell \rangle$  in the remaining denominators, since the master cuts contain at most one:

$$\begin{aligned} \frac{\prod_{j=1}^{n-1} [a_j \tilde{\lambda}]}{\langle \lambda | K | \tilde{\lambda} \rangle^n \langle \lambda | Q | \tilde{\lambda} \rangle} &= \frac{\prod_{j=1}^{n-1} [a_j | Q | \tilde{\lambda} \rangle]}{\langle \lambda | K Q | \lambda \rangle^{n-1}} \frac{1}{\langle \lambda | K | \tilde{\lambda} \rangle \langle \lambda | Q | \tilde{\lambda} \rangle} \\ &\quad - \sum_{p=0}^{n-2} \frac{\left( \prod_{j=1}^{n-p-2} [a_j | Q | \lambda] \right) [a_{n-p-1} | K | \lambda] \left( \prod_{t=n-p}^{n-1} [a_t \tilde{\lambda}] \right)}{\langle \lambda | K | \tilde{\lambda} \rangle^{p+2} \langle \lambda | K Q | \lambda \rangle^{n-p-1}}. \end{aligned} \quad (41)$$

Power-counting arguments ensure that enough appearances of  $\tilde{\lambda}$  in the numerator to implement these identities as often as necessary.

It remains to implement the residue theorem, with the help of a generalized version of the differentiation identity (29) and the careful treatment of higher-multiplicity poles arising in the factor  $\langle \lambda | KQ | \lambda \rangle$ .

The procedure can be performed in generality. Formulas for the coefficients derived from ordinary unitarity cuts have been given in [10, 11, 12, 13]. These references include  $D$ -dimensional versions of these formulas, with possible scalar masses. Similar formulas based on generalized cuts have been given in [8, 14, 15].

## 6 The OPP algorithm (numerical)

The OPP (Ossola, Papadopoulos, Pittau) algorithm [16] for finding the coefficients of master integrals is a procedure based on numerical solutions to the on-shell conditions of generalized unitarity cuts. It is carried out at the *integrand* level.

In addition to the integrands for the scalar box, triangle, bubble, and tadpole, there are “spurious” terms that integrate to zero. These have been thoroughly classified. They have no more than four denominator factors but have nontrivial tensor structure in the numerator. Indeed, two of the three steps in integral reduction can be carried out at the integrand level as described above. It is the argument based on Lorentz covariance (step 2) that breaks down for integrands.

If a general integrand is written in the form  $I = N(\ell)/(D_0 D_1 \cdots D_{n-1})$ , where  $D_i = (\ell - K_i)^2 - M_i^2$ , the OPP expansion is of the form

$$\begin{aligned} I(\ell) = & \sum_i [a(i) + \tilde{a}(\ell; i)] I^{(i)} + \sum_{i < j} [b(i, j) + \tilde{b}(\ell; i, j)] I^{(i, j)} \\ & + \sum_{i < j < r} [c(i, j, r) + \tilde{c}(\ell; i, j, r)] I^{(i, j, r)} + \sum_{i < j < r < s} [d(i, j, r, s) + \tilde{d}(\ell; i, j, r, s)] I^{(i, j, r, s)}. \end{aligned}$$

Here,  $a(i), b(i, j), c(i, j, r), d(i, j, r, s)$  are the coefficients of the master integrals containing the specified propagators. The coefficients with tildes and  $\ell$ -dependence are the spurious terms, listed explicitly in [16]. There is just one spurious term for the box, but several for the lower-point integrands.

The procedure is triangular. Start with the box coefficients. For example, solve for  $d(0, 1, 2, 3)$  by multiplying through by  $D_0 D_1 D_2 D_3$  and plugging in the numerical solutions  $\ell_{0123}^\pm$  to the equations

$$0 = D_0(\ell) = D_1(\ell) = D_2(\ell) = D_3(\ell). \quad (42)$$

Since the only term without explicit vanishing factors is the term for the box  $(0, 1, 2, 3)$ , the OPP expansion collapses to the equation

$$I(\ell_{0123}^\pm) = d(0, 1, 2, 3) + \tilde{d}(\ell_{0123}^\pm; 0, 1, 2, 3). \quad (43)$$

Explicitly, the four-point spurious term is  $\tilde{d}(\ell; i, j, r, s) = \epsilon(\ell, K_j, K_r, K_s)$ . Since there are two solutions to (42), we can solve for both the true box coefficient

$d(0, 1, 2, 3)$  and the spurious box coefficient. Indeed, up to this point it is exactly like the quadruple cut.

The numerical benefit of the OPP algorithm appears in the following steps, where we proceed to solve triples and duples of on-shell conditions. Analytically, there are whole families of solutions, but numerically, one can just choose as many different solutions from these families as needed to solve for all the spurious coefficients.

## 7 $D$ -dimensional unitarity / rational terms

In closing, I will merely mention various current approaches to computing the rational terms in (15).

- Keep  $\epsilon$  dependence and perform cuts strictly in  $D$  dimensions, so that there are no separate rational terms. Analytic formulas are given among the references previously mentioned, [11, 12, 13, 15]. Numerically, it can be convenient to sample *integer* values of  $D$ ; two values beyond  $D = 4$  are enough [17]. With integer  $D$ , gamma-matrix algebra is straightforward, and the tree-level input can be generated efficiently from Berends-Giele or similar off-shell recursions.
- Rational terms can be generated from on-shell recursion, since they have no branch cuts. However, their pole structure is significantly more complicated than at tree level. See [19, 20] for reviews.
- Feynman diagrams can be brought in to complement unitarity-cut techniques. If only the rational part is needed, the diagram analysis simplifies a lot. Examples of this approach appear in [21, 22].
- The rational component of the OPP method (due to DGPP rather than OPP [18]) separates the rational terms into two parts. One is computed from  $D$ -dimensional cuts, while the other is obtained from special additional Feynman rules.

## 8 Exercise

Consider the  $2 \rightarrow 2$  scattering of photons (“light by light”) mediated by a virtual electron loop.

1. List the box integrals. (Note: There is no color ordering.)
2. What symmetries relate the various coefficients of the helicity amplitudes?
3. Compute the box coefficients.



The tree amplitudes for the scattering of photons with an electron-positron pair vanish when all photons have like helicity. For one photon of opposite helicity, the MHV amplitude is given by

$$A(\bar{e}^+, e^-, 1^+, 2^+, \dots, i^-, \dots, n^+) = 2^{\frac{n}{2}} e^n \frac{\langle f \bar{f} \rangle^{n-2} \langle f i \rangle^3 \langle \bar{f} i \rangle}{\prod_{k=1}^n \langle f k \rangle \langle \bar{f} k \rangle}.$$

*Additional comments.* The full amplitude includes scalar triangle and bubble integrals and rational terms as well. For  $n$ -photon scattering mediated by a fermion loop, rational terms and bubbles are absent for  $n > 4$ , and triangles are absent for  $n > 6$ . The amplitudes vanish for odd values of  $n$  by Furry's theorem.

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