Exact results for twist operators in planar $\mathcal{N} = 4$ SYM

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Plan of the talk

- Very brief introduction to the Bethe ansatz
  - The $\mathfrak{sl}(2)$ subsector
    - General properties
    - Higher charges
    - Exact solutions : twist-two and three
  - Non-Linear Integral Equation
    - Motivation and basic ideas
    - One-loop NLIE for $\mathfrak{sl}(2)$ operators
    - Non-Linear Beisert-Eden-Staudacher equation
    - Finite size corrections
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The symmetry algebra of $\mathcal{N} = 4$ SYM is $\mathfrak{psu}(2, 2|4)$.

A convenient choice of the Dynkin diagram is [Beisert, Staudacher '05]

```
\begin{center}
\begin{tikzpicture}
\vertex (1) at (-2,0) [draw] ;
\vertex (2) at (0,0) [draw] ;
\vertex (3) at (2,0) [draw] ;
\vertex (4) at (4,0) [draw] ;
\vertex (5) at (6,0) [draw] ;
\vertex (6) at (8,0) [draw] ;
\vertex (7) at (10,0) [draw] ;
\draw (1) to (2);
\draw (3) to (2);
\draw (5) to (4);
\draw (7) to (6);
\end{tikzpicture}
\end{center}
```

Gauge-invariant operators belong to the unitary representations of the above-mentioned algebra.

The theory in the planar limit $N \to \infty$ is believed to be integrable. As a consequence the problem of finding anomalous dimensions of the operators can be solved without computing a single Feynman diagram!
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![Dynkin diagram](image)

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Asymptotic All-Loop Bethe Equations

\[ 1 = \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} + \frac{i}{2}}{u_{1,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{1 - g^2/\ x_{1,k}x_{4,j}^+}{1 - g^2/\ x_{1,k}x_{4,j}^-}, \]

\[ 1 = \prod_{j=1}^{K_2} \frac{u_{2,k} - u_{2,j} - i}{u_{2,k} - u_{2,j} + i} \prod_{j=1}^{K_3} \frac{u_{2,k} - u_{3,j} + \frac{i}{2}}{u_{2,k} - u_{3,j} - \frac{i}{2}} \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} + \frac{i}{2}}{u_{2,k} - u_{1,j} - \frac{i}{2}}, \]

\[ 1 = \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + \frac{i}{2}}{u_{3,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k} - x_{4,j}^+}{x_{3,k} - x_{4,j}^-}, \]

\[ 1 = \left( \frac{x_{4,k}^-}{x_{4,k}^+} \right)^{L \prod_{j=1}^{K_4} \left( \frac{u_{4,k} - u_{4,j} + i}{u_{4,k} - u_{4,j} - i} \sigma^2(x_{4,k}, x_{4,j}) \right)} \]

\[ \times \prod_{j=1}^{K_1} \frac{1 - g^2/\ x_{4,k}^-x_{1,j}}{1 - g^2/\ x_{4,k}^+x_{1,j}} \prod_{j=1}^{K_3} \frac{x_{4,k}^- - x_{3,j}}{x_{4,k}^+ - x_{3,j}} \prod_{j=1}^{K_5} \frac{x_{4,k}^- - x_{5,j}}{x_{4,k}^+ - x_{5,j}} \prod_{j=1}^{K_7} \frac{1 - g^2/\ x_{4,k}^-x_{7,j}}{1 - g^2/\ x_{4,k}^+x_{7,j}}, \]

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Where [Beisert, Dippel, Staudacher ’04]

\[ x(u) = \frac{u}{2} \left( 1 + \sqrt{1 - \frac{4g^2}{u^2}} \right), \quad x^\pm = x(u \pm \frac{i}{2}). \]

The higher charges are given by

\[ Q_r = \frac{i}{r-1} \sum_{j=1}^{K_4} \left( \frac{1}{(x^+(u_j))^{r-1}} - \frac{1}{(x^-(u_j))^{r-1}} \right) \]

Anomalous dimension corresponds to the \( Q_2 \) charge

\[ \gamma(g) = 2g^2 Q_2 \]
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The $\mathfrak{sl}(2)$ subsector

- The field content:

$$\mathcal{O} = \text{Tr} \left( \mathcal{D}^M \mathcal{Z}^L \right) + \ldots ,$$

where $\mathcal{D} = \mathcal{D}_1 + i \mathcal{D}_2$ and $D_\mu = \partial_\mu + iA_\mu$.

- Asymptotic Bethe equations [Staudacher ’04; Beisert, Staudacher ’05]

$$\left( \frac{x_k^+}{x_k^-} \right)^L = \prod_{j \neq k}^M \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1 - g^2 / x_k^+ x_j^-}{1 - g^2 / x_k^- x_j^+} \sigma^2(u_k, u_j).$$

- These equations are valid only up to the wrapping order $\mathcal{O}(g^{2L+4})$.

- At one-loop this is a spin $-\frac{1}{2}$ non-compact magnet.
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All Bethe roots in this sector are real at weak coupling.

For a fixed value of $L$ the corresponding anomalous dimension of the ground state has the following asymptotic behavior

$$\gamma(g) = 2g^2 Q_2 = f(g) \log M + C(g, L) + O\left(\frac{1}{M}\right)$$

at large values of $M$.

The scaling function is conjectured to be $L$ independent.

At weak coupling it can be found from the solution of the BES equation [Beisert, Eden, Staudacher '06]

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The quantity $\hat{\sigma}(t)$ is the solution of

$$\hat{\sigma}(t) = \frac{t}{e^t - 1} \left( K(2g t, 0) - 4g^2 \int_0^\infty dt' K(2g t, 2g t') \hat{\sigma}(t') \right)$$

It has an interpretation as a fluctuation density [Eden, Staudacher '06]

$$\rho(u) = \rho_0(u) - 8g^2 \frac{\log(M)}{M} \sigma(u)$$

At strong coupling string theory predicts [Gubser, Klebanov, Polyakov '02], [Frolov, Tseytlin, '02]

$$f(g) = 4g - \frac{3 \log 2}{\pi} + O\left(\frac{1}{g}\right)$$

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Higher charges

- What about the higher charges?

- Starting from the two-loop order they also scale logarithmically

\[ Q_r = f_r(g) \log(M) + C_r(g, L) + \mathcal{O}\left(\frac{1}{M}\right) \]

- At weak coupling these scaling functions are \( L \) independent and also obey the *transcendentality principle*, for example

\[
\begin{align*}
f_4(g) &= 16 \zeta(4) g^4 - 16 \left(2 \zeta(2) \zeta(4) + 15 \zeta(6)\right) g^6 \\
&\quad + 32 \left(2 \zeta(2)^2 \zeta(4) + 6 \zeta(4)^2 + 4 \zeta(3) \zeta(5)\right) g^8 + \ldots
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Higher charges at strong coupling

The leading density at strong coupling was found in [Alday, Arutyunov, Benna, Eden, Klebanov ’07].

Integrating it over the charge densities gives

\[ f_r(g) = \left( \frac{1}{g} \right)^{r-1} \left( \frac{\Gamma\left[\frac{r-1}{2}\right]}{\Gamma\left[\frac{r}{2}\right] \Gamma\left[\frac{1}{2}\right]} - \frac{4}{\pi} \frac{3F_2\left(\frac{3}{2}, \frac{3}{2} - \frac{r}{2}, \frac{1}{2} + \frac{r}{2}; \frac{5}{2} - \frac{r}{2}, \frac{3}{2} + \frac{r}{2}; 1\right)}{(r^2 - 2r - 3)} \right). \]

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At one-loop Bethe equations are equivalent to the polynomial solution of the Baxter equation

\[
\left(u + \frac{i}{2}\right)^L Q(u + i) + \left(u - \frac{i}{2}\right)^L Q(u - i) = t(u) Q(u),
\]

where

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t(u) = 2u^L + \sum_{i=2}^{L} \tilde{q}_i u^{L-i}.
\]

For \(L = 2\) this equation can be exactly solved [Virginia Dippel, unpublished]

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Q_2(u) = {}_3\!F_2\left(-M, M + 1, \frac{1}{2} + iu, ; 1, 1; 1\right).
\]
Exact solutions

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Twist-three at one-loop

Surprisingly, one can also solve the Baxter equation for the ground state of $L = 3$ [Kotikov, Lipatov, A.R., Staudacher, Velizhanin, '07], [Beccaria, '07]

$$Q_3(u) = {}_4F_3\left(-\frac{M}{2}, \frac{M}{2} + 1, \frac{1}{2} + iu, \frac{1}{2} - iu; 1, 1, 1; 1\right).$$

Defining the nested harmonic sums:

$$S_a(M) = \sum_{i=1}^{M} \frac{(\text{sgn}(a))^i}{i|a|}, \quad S_{a_1,...,a_n}(M) = \sum_{i=1}^{M} \frac{(\text{sgn}(a_1))^i}{i|a_1|} S_{a_2,...,a_n}(i),$$

one easily derives the corresponding anomalous dimension

$$\gamma_2(M) = 8 S_1 \left(\frac{M}{2}\right).$$
Surprisingly, one can also solve the Baxter equation for the ground state of $L = 3$ [Kotikov, Lipatov, A.R., Staudacher, Velizhanin, '07], [Beccaria, '07]

$$Q_3(u) = 4 F_3 \left( -\frac{M}{2}, \frac{M}{2} + 1, \frac{1}{2} + iu, \frac{1}{2} - iu, ; 1, 1, 1; 1 \right).$$

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\gamma_2(M) = 8 \, S_1 \left( \frac{M}{2} \right)
\]
Twist-three at higher loops

At higher-loops one can derive

$$\frac{\gamma_{ABA}^4(M)}{4} = -2S_3 - 4S_1 S_2,$$

$$\frac{\gamma_{ABA}^6(M)}{8} = 2S_2 S_3 + S_5 + 4S_{3,2} + 4S_{4,1} - 8S_{3,1,1} + S_1 \left(4S_2^2 + 2S_4 + 8S_{3,1}\right)$$
\[
\frac{\gamma_8^{ABA}(M)}{16} = S_1^3 \left( \frac{40}{3} S_4 - \frac{32}{3} S_{3,1} \right) + S_1^2 \left( 20 S_5 - 40 S_{3,2} - 56 S_{4,1} + 64 S_{3,1,1} \right) \\
+ S_1 \left( 7 S_6 + 8 S_{2,4} - 24 S_{3,3} - 56 S_{4,2} - 40 S_{5,1} - 24 S_{2,2,2} - 16 S_{2,3,1} + 88 S_{3,1,2} + 88 S_{3,2,1} + 120 S_{4,1,1} - 192 S_{3,1,1,1} - 8 \zeta(3) S_3 \right) - \frac{56}{3} S_3 S_4 - \frac{107}{6} S_7 + 3 S_{2,5} + \frac{41}{3} S_{3,4} + \frac{1}{3} S_{4,3} \\
- 17 S_{5,2} - \frac{20}{3} S_{6,1} - 4 S_{2,2,3} - 8 S_{2,3,2} - 4 S_{2,4,1} + \frac{104}{3} S_{3,1,3} + 52 S_{3,2,2} + \frac{88}{3} S_{3,3,1} + 60 S_{4,1,2} + 60 S_{4,2,1} + 40 S_{5,1,1} + 8 S_{2,3,1,1} - 120 S_{3,1,1,2} - 120 S_{3,1,2,1} - 120 S_{3,2,1,1} - 128 S_{4,1,1,1} + 256 S_{3,1,1,1,1}
\]
It is not known how to derive these formulas (and similar for the twist-two case) analytically from the Bethe ansatz.

There is a detour, however, when one assumes the Kotikov-Lipatov transcendentality principle. [Kotikov, Lipatov '02]

Curiously, only positive indices of the harmonic sums appear.

It would be interesting to investigate in general when the Bethe equations are exactly solvable.
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Non-linear Integral Equation

Motivation

- A finite $M$ NLBES equation
- Finite size effects
  - e.g. $O(M^0)$ corrections
  - A tool for deriving analytically the anomalous dimension in terms of harmonic sums for $L = 2, 3$?
- Strong-coupling
- Considering different limits
- An independent derivation of the BES equation
- Relation to the BFKL physics / repairing the asymptotic Bethe equations.
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We consider the ground state at fixed $L$ and $M$.

The basic step towards constructing the NLIE is to introduce the complementary set of solutions of the $\mathfrak{sl}(2)$ Bethe equations, termed holes.

The dynamics of the holes is determined by

$$t_g(u_h) = 0.$$ 

A proposal for the asymptotic Baxter equation was made in [Belitsky, Korchemsky, Müller '06].

There are precisely $L$ solutions of the above equation and therefore they are to be identified with the $\mathcal{Z}$ fields.
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\[ \text{Tr} \left( D^2 Z DZ D^3 Z DZ DZ DZ DZ DZ \ldots \right) \]
Two out of these $L$ holes are special. [Belitsky, Korchemsky, Gorsky '06]

They scale as

$$u_h^1 = -u_h^2 \approx \frac{M}{\sqrt{2}}.$$ 

They are responsible for the logarithmic scaling of the anomalous dimension and for the universality of the corresponding scaling function.

The remaining holes contribute starting from the $O \left( \frac{1}{M} \right)$ order.

A closed formula for the hole rapidities is known only in the exceptional cases ($L = 2, 3$), see below.
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One-loop NLIE

- For the one-loop \( \mathfrak{sl}(2) \) Bethe equations one defines the counting function as

\[
Z(u) = L\phi(u, 1/2) + \sum_{k=1}^{M} \phi(u - u_k) \quad \text{where} \quad \phi(u, \xi) = i \log \left( \frac{i\xi + u}{i\xi - u} \right).
\]

- Bethe roots and the holes satisfy

\[
e^{iZ(u_k)} = (-1)^{\delta - 1} \quad k = 1, \ldots, M + L
\]

- The following identity is crucial in constructing the NLIE [Feverati, Fioravanti, Grinza, Rossi '06]

\[
\sum_{k=1}^{M} f(u_k) = -\int_{-\infty}^{\infty} \frac{dx}{2\pi} f'(x) Z(x) + \\
+ \int_{-\infty}^{\infty} \frac{dx}{\pi} f'(x) \text{Im} \ln \left[ 1 + (-1)^{\delta} e^{iZ(x+i0)} \right] - \sum_{h=1}^{L} f(x_h).
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Using the above identity for $Z(u)$ one gets the non-linear integral equation

$$
Z(u) = iL \log \frac{\Gamma(1/2 + iu)}{\Gamma(1/2 - iu)} + \sum_{j=1}^{L} i \log \frac{\Gamma(-i(u - u_{h}^{(j)}))}{\Gamma(i(u - u_{h}^{(j)}))} \\
+ \lim_{\alpha \to \infty} \int_{-\alpha}^{\alpha} \frac{dv}{\pi} \frac{i}{du} \log \frac{\Gamma(-i(u - v))}{\Gamma(i(u - v))} \text{Im} \log \left[ 1 + (-1)^{\delta} e^{iZ(v+i0)} \right],
$$

The conserved charges are related to $Z(u)$ by

$$
Q_p = - \int \frac{dv}{2\pi} q'_p(v) Z(v) - \sum_{j=1}^{L} q(p)(u_{h}^{(j)}) \\
+ \int \frac{dv}{\pi} q'_p(v) \text{Im} \log \left[ 1 + (-1)^{\delta} e^{iZ(v+i0)} \right]
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In particular, one finds for the $Q_2$ charge

$$Q_2 = 2\gamma L + \sum_{j=1}^{L} \left\{ \psi(1/2 + i\eta_{j}) + \psi(1/2 - i\eta_{j}) \right\}$$

$$+ \int_{-\infty}^{\infty} \frac{dv}{\pi} \frac{d^2}{dv^2} \left( \log \frac{\Gamma (1/2 + iv)}{\Gamma (1/2 - iv)} \right) \text{Im} \log \left[ 1 + (-1)^{\delta} e^{iZ(v+i0)} \right]$$

$M$ dependence is hidden in the hole roots. For example, for $L = 2$

$$\eta_{1}^{1} = -\eta_{2}^{2} = \sqrt{\frac{1}{2} \left( M^2 + M + \frac{1}{2} \right)}$$

Terms involving

$$L(u) = \text{Im} \log \left[ 1 + (-1)^{\delta} e^{iZ(u+i0)} \right]$$

contribute starting from the $\mathcal{O} \left( \frac{1}{M} \right)$ order.
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$$+ \int_{-∞}^{∞} \frac{dv}{\pi} \frac{id^2}{dv^2} \left( \log \frac{Γ (1/2 + iv)}{Γ (1/2 - iv)} \right) \text{Im} \log \left[ 1 + (-1)^δ e^{iZ(v+i0)} \right]$$

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$$

$$
+ \int_{-\infty}^{\infty} \frac{dv}{\pi} i \frac{d^2}{dv^2} \left( \log\frac{\Gamma\left(\frac{1}{2} + iv\right)}{\Gamma\left(\frac{1}{2} - iv\right)} \right) \text{Im} \log \left[ 1 + (-1)^{\delta} e^{iZ(v+i0)} \right]
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At higher loops one defines

\[
Z(u) = \text{Li} \log \frac{x(i/2 + u)}{x(i/2 - u)} + i \sum_{k=1}^{M} \log \frac{i + u - u_k}{i - (u - u_k)}
\]

\[-2i \sum_{k=1}^{M} \log \frac{1 - \frac{g^2}{x^+ x_k^-}}{1 - \frac{g^2}{x^- x_k^+}} - i \sum_{k=1}^{M} \log \sigma^2(u, u_k),\]

The corresponding non-linear integral equation reads...
NLIE at higher loops

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The corresponding non-linear integral equation reads
\[ Z(u) = iL \log \frac{x(i/2 + u)}{x(i/2 - u)} + \int_{-\infty}^{\infty} \frac{dv}{2\pi} \phi'(u - v, 1)Z(v) \]

\[ - \sum_{j=1}^{L} \phi(u - u_{h}^{(j)}, 1) - \int_{-\infty}^{\infty} \frac{dv}{\pi} \phi'(u - v, 1)\text{Im} \log \left[ 1 + (-1)^{\delta} e^{iZ(v+i0)} \right] \]

\[ \int_{-\infty}^{\infty} \frac{dv}{2\pi} \left( -2i \frac{d}{dv} \log \frac{1 + \frac{g^2}{x(i/2+u)x(i/2-v)}}{1 + \frac{g^2}{x(i/2-u)x(i/2+v)}} \right) Z(v) \]

\[ - \sum_{j=1}^{L} \left( -2i \log \frac{1 + \frac{g^2}{x(i/2+u)x(i/2-u_{h}^{(j)})}}{1 + \frac{g^2}{x(i/2-u)x(i/2+u_{h}^{(j)})}} \right) \]

\[ + \int_{-\infty}^{\infty} \frac{dv}{\pi} \left( -2i \frac{d}{dv} \log \frac{1 + \frac{g^2}{x(i/2+u)x(i/2-v)}}{1 + \frac{g^2}{x(i/2-u)x(i/2+v)}} \right) \text{Im} \log \left[ 1 + (-1)^{\delta} e^{iZ(v+i0)} \right] \]

\[ + 2 \sum_{r,s} \beta_{r,s} (q_{r}(u)Q_{s} - q_{s}(u)Q_{r}) \]
The conserved charges are related to $Z(u)$ in a similar way, as in the one-loop case

$$Q_p = -\int \frac{dv}{2\pi} q'_p(v)Z(v) - \sum_{j=1}^{L} q_p(u^{(j)}_h)$$

$$+ \int \frac{dv}{\pi} q'_p(v) \text{Im} \log \left[ 1 + (-1)^{\delta} e^{iZ(v+i0)} \right]$$

and defining

$$Q_p = g^{p-1} i^{p+2} \frac{i}{p-1} \sum_k \left( \frac{1}{(x^+(u_k))^{p-1}} - \frac{(-1)^p}{(x^-(u_k))^{p-1}} \right),$$

one can transform the NLIE equation to
\[ Q_p = 2L \int_0^\infty dt \frac{J_0(2g t)J_{p-1}(2g t)}{t(e^t - 1)} \]

\[- \sum_{j=1}^L \int_0^\infty dt \frac{J_{p-1}(2g t)}{t(1 - e^{-t})} \left( e^{-it(u^j_h - i/2)} + e^{it(u^j_h + i/2)} \right) \]

\[- 2 \int_0^\infty \frac{dt}{\pi} \frac{iJ_{p-1}(2g t)e^{-t/2}}{1 - e^{-t}} \hat{L}(t) + \sum_{r=1}^\infty r(-1)^{r+1}Q_{r+1} d_{r+1,p} \]

\[- \sum_{r=1}^\infty \sum_{s=r+1}^\infty 8 \left( 2r(2s - 1) d_{2r+1,2s} d_{p,2r+1} Q_{2s} \right. \]

\[+ \left. (2r - 1)(2s - 2) d_{2r,2s-1} d_{p,2s-1} Q_{2r} \right) . \]

- \( d_{r,s} \) are given by

\[ d_{r,s}(g) = \int_0^\infty dt \frac{J_{r-1}(2g t)J_{s-1}(2g t)}{t(e^t - 1)} . \]
Non-linear BES equation

In the Fourier space the NLIE generalizes the Beisert-Eden-Staudacher equation to finite values of $M$

$$
\hat{Z}(t) = \frac{2 \pi L e^\frac{t}{2}}{it(e^t - 1)} J_0(2g t) - \sum_{j=1}^{L} \frac{2 \pi \cos \left( t u_h^{(j)} \right)}{it(e^t - 1)} - \frac{2}{e^t - 1} \hat{L}(t)
$$

$$
+ 8 g^2 \frac{e^\frac{t}{2}}{e^t - 1} \int_0^{\infty} dt' e^{-t'^2} K(2g t, 2g t') \left( t' \hat{L}(t') \right)
$$

$$
- \frac{\pi}{i} \sum_{j=1}^{L} \cos \left( t' u_h^{(j)} \right)
$$

$$
- 4 g^2 \frac{e^\frac{t}{2}}{e^t - 1} \int_0^{\infty} dt' e^{-t'^2} t' K(2g t, 2g t') \hat{Z}(t')
$$

This equation is completely equivalent to the discrete (asymptotic) Bethe equations!
Non-linear BES equation

In the Fourier space the NLIE generalizes the Beisert-Eden-Staudacher equation to finite values of $M$

\[
\hat{Z}(t) = \frac{2 \pi L e^{\frac{t}{2}}}{i t (e^t - 1)} J_0(2 g t) - \sum_{j=1}^{L} \frac{2 \pi \cos \left( t u_h^{(j)} \right)}{i t (e^t - 1)} - \frac{2}{e^t - 1} \hat{L}(t)
\]

\[
+ 8 g^2 \frac{e^{\frac{t}{2}}}{e^t - 1} \int_0^{\infty} dt' e^{-\frac{t'}{2}} K(2 g t, 2 g t') \left( t' \hat{L}(t') \right)
\]

\[
- \frac{\pi}{i} \sum_{j=1}^{L} \cos \left( t' u_h^{(j)} \right)
\]

\[
- 4 g^2 \frac{e^{\frac{t}{2}}}{e^t - 1} \int_0^{\infty} dt' e^{-\frac{t'}{2}} t' K(2 g t, 2 g t') \hat{Z}(t')
\]

This equation is completely equivalent to the discrete (asymptotic) Bethe equations!
It is a simple application of the above-presented equations to calculate the $O(M^0)$ to the anomalous dimension at arbitrary loop order

$$C(g, L) = \gamma f(g) - 8(7 - 2L)\zeta(3)g^4 + 8\left(\frac{4 - L}{3}\pi^2\zeta(3)\right)$$

$$+ (62 - 21L)\zeta(5)g^6 - \frac{8}{15}\left((13 - 3L)\pi^4\zeta(3)\right)$$

$$+ 5(32 - 11L)\pi^2\zeta(5) + 75(127 - 46L)\zeta(7)\right)g^8$$

$$+ 32\left(\frac{4}{945}(49 - 11L)\pi^6\zeta(3) - (14 - 4L)\zeta(3)^3\right)$$

$$+ \frac{1}{180}(310 - 103L)\pi^4\zeta(5) + \left(\frac{5}{12}(64 - 5L)\pi^2\zeta(7)\right)$$

$$+ \frac{49}{4}(146 - 55L)\zeta(9)\right)g^{10} + \ldots$$
BES from NLBES

In the large $M$ limit the NLBES simplifies to

\[
\hat{Z}(t') = 8 \pi i g^2 \frac{e^{\frac{t}{2}}}{e^t - 1} K(2 g t, 0) \log(M) \\
- 4 g^2 \frac{e^{\frac{t}{2}}}{e^t - 1} \int_0^\infty dt' e^{-\frac{t'}{2}} t' K(2 g t, 2 g t') \hat{Z}(t')
\]

Under the identification

\[
\hat{Z}(t) = 8 \pi i g^2 \frac{e^{\frac{t}{2}}}{e^t - 1} \frac{\sigma(t)}{t} \log(M)
\]

one recovers the BES equation

\[
\hat{\sigma}(t) = \frac{t}{e^t - 1} \left( K(2 g t, 0) - 4 g^2 \int_0^\infty dt' K(2 g t, 2 g t') \hat{\sigma}(t') \right)
\]
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$$
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- 4 g^2 \frac{e^{t/2}}{e^t - 1} \int_0^\infty \! dt' e^{-t'/2} t' K(2g t, 2g t') \hat{Z}(t')
$$

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The above-presented derivation of the BES equation is qualitatively different from the original derivation.

- There is no splitting into the one-loop density and the fluctuation density.
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Conclusions

- All the conserved charges of the $\mathfrak{sl}(2)$ subsector scale as $\log(M)$ at large values of $M$. This offers the possibility to compare the whole integrable structure on both sides of planar AdS/CFT.

- In some special cases the anomalous dimension can be explicitly found as function of $M$. It would be interesting to investigate what are the precise conditions for such hyperintegrability.

- The Non-Linear Beisert-Eden-Staudacher equation for the ground states of $\mathfrak{sl}(2)$ was derived.

- This non-linear equation is completely equivalent to the discrete Bethe ansatz.
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More conclusions

- With the use of this equation one can derive up to the wrapping order the $O(M^0)$ corrections to the anomalous dimension of twist operators.

- It offers the possibility to derive independently the Beisert-Eden-Staudacher equation.

- It would be interesting to generalize this method to
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