Cours de physique théorique de Saclay

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Vendredi 17/02/2023, 10:00-12:00

Orme des Merisiers Salle Claude Itzykson, Bât. 774

Rough paths for physicists

Michel Bauer

Rough path theory emerged in the 1990's motivated by practical and conceptual questions. The general context is that of controlled differential equations, for example $dU_t = f(U_t)dX_t$ where X_t is a known property of a system, changing with time, and U_t is an unknown whose evolution is to be computed via the equation. The emphasis is on the regularity of X_t as a function of time, and its consequences on that of U_t, making the necessary assumptions on f.
If X_t is smooth, we are on familiar grounds, but what if X_t, though continuous, has little regularity? In that case, it is the rule rather than the exception that Euler's discretization $U_{t_{n+1}} = U_{t_n} + f(U_{t_n})(X_{t_{n+1}} - X_{t_n})$ does not converge when the mesh goes to 0. Approaching X_t more and more closely by smooth functions $X_t^{(n)}$, and seeing if the corresponding $U_t^{(n)}$s get close to something, has its own difficulties. What is the right notion of "close"? Does $\lim_{n\to\infty} U_t^{(n)}$, when it exists, depend on the approximation sequence $X_t^{(n)}$ and how?
Rough path theory gives an answer to these questions in a form suggestive of deep relations to physics (power counting, counterterms, etc) that I will try to explain. The short answer is that depending on the regularity of X_t, making unambiguous sense of the equation $dU_t = f(U_t)dX_t$ requires to supplement X_t with new data which roughly (!) speaking replace some undefined integrals by a formal, yet concrete, mathematical object: a rough path structure.

Plan:
1. Motivations. The sewing lemma and Young integrals.
2. Rough paths, combinatorics and regularity. A waltz with Brownian motion.
3. Controlled rough paths, rough integrals and rough differential equations.
Rough integrals versus stochastic integrals, the Itō stochastic area.
4. Thermalization of a particle in a magnetic field. Outlook.

Course website: https://courses.ipht.fr/?q=en/node/309

Videoconference and in person in Salle Itzykson, IPhT

Pour toute information, contacter ipht-lectures@cea.fr