Topological order in quantum dimer models

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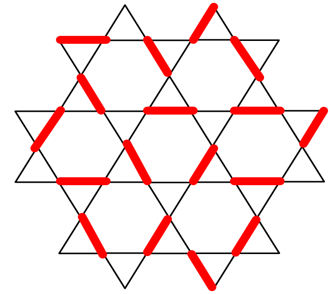
Outline

- What is quantum dimer model ?
- A toy model on the kagome lattice :
  - exact mapping to Kitaev's toric code
  - dimer liquid with Z_2 topological order

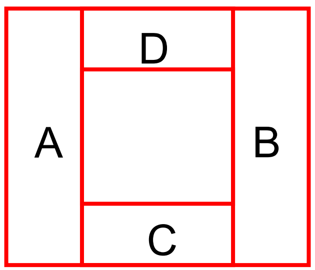


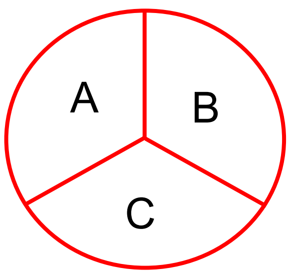
- Liquid phase of the triangular lattice quantum dimer model (RK point):
  - 4 degenerate ground-state on a torus
  - Numerics showing that these state are locally undistinguishable
  - Computation of the topological entanglement entropy :

  \- Kitaev-Preskill and Levin-Wen constructions

  \- Modified construction to extract the topological entanglement entropy

  $\Rightarrow \gamma = 1.00(2)\log(2)$

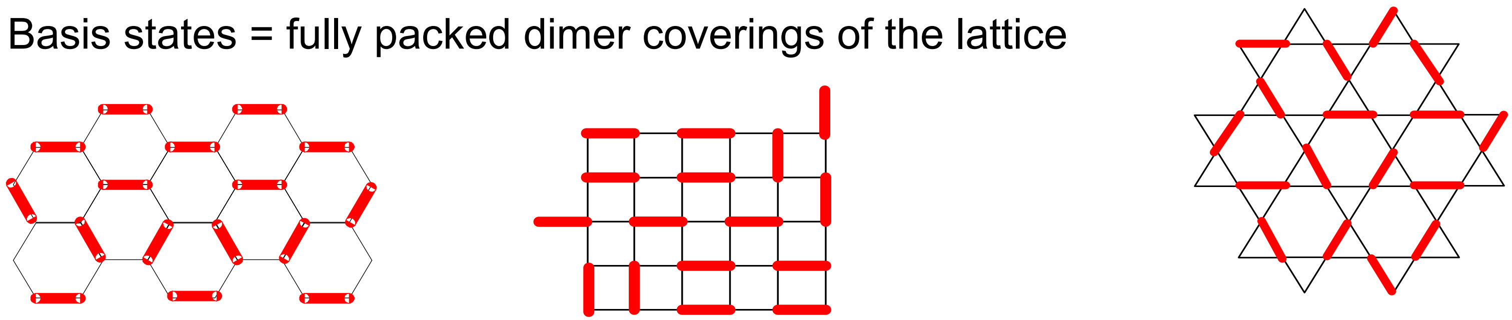




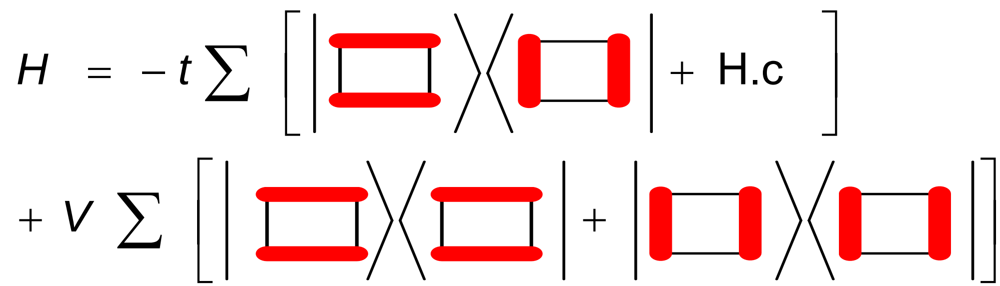
What is a quantum dimer model ?

Quantum dimer models

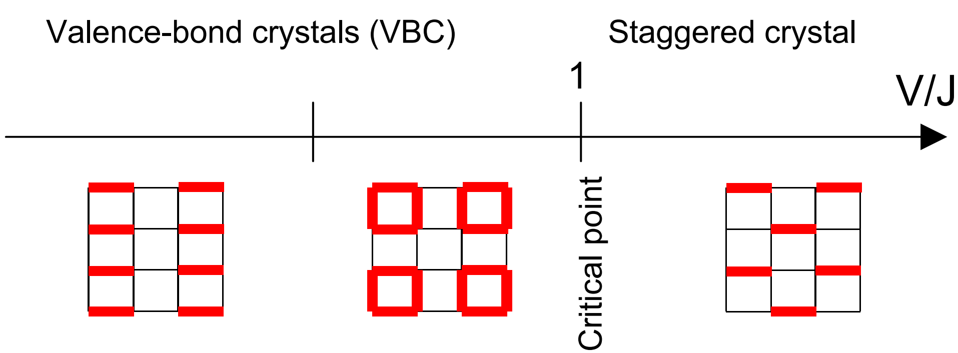
- Basis states = fully packed dimer coverings of the lattice



- Introduce some simple dynamics



- Examples of phase diagrams

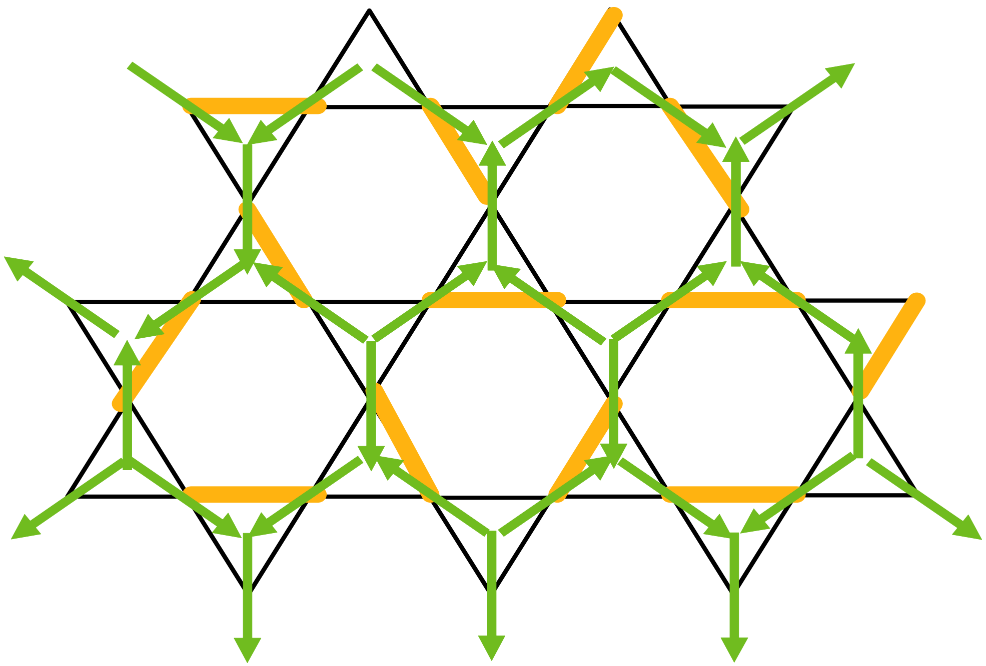


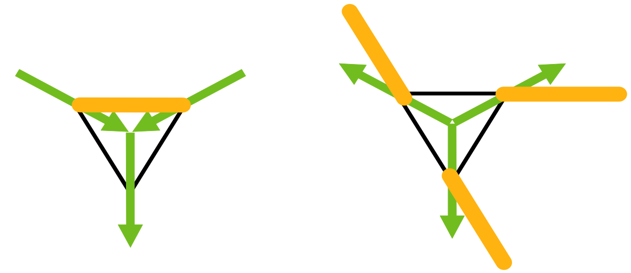
Rokhsar & Kivelson, Phys. Rev. Lett. (1988).
Syljuasen, Phys. Rev. B 73, 245105 (2006) + refs. therein

An exactly solvable quantum dimer model on the kagome lattice

Dimer coverings of the *kagome* lattice

On a lattice made of corner-sharing triangles (such as kagome), dimer coverings are easily represented with **arrows** :





Constraint

Number of incoming arrows must be *even* on every triangle

N_{in}=0 or 2

There is a one-to-one correspondence between

i) configurations of arrows with N_{in}=even everywhere and

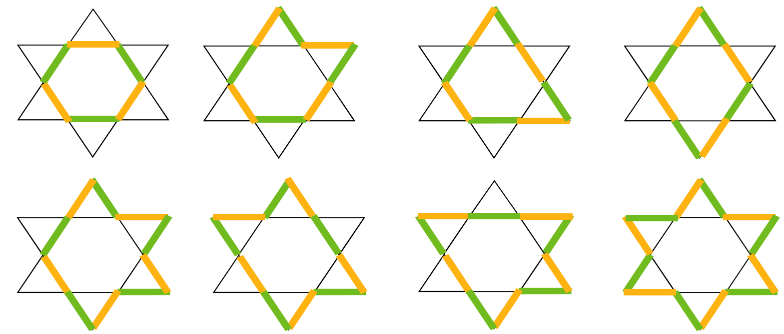
ii) hard-core dimer coverings

Exactly solvable dimer (toy) model

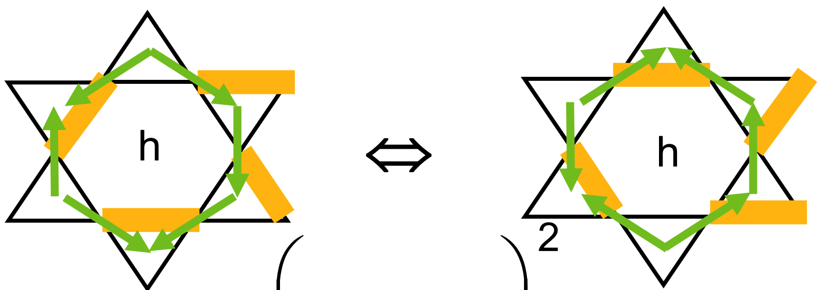
τ_i^z = Flips the arrow ***i***

$$\prod_{\text{closed loop}} \tau_i^z = \text{Physical dimer move (preserves the constraints)}$$

GM, Serban & Pasquier, PRL 2002



32 different loops when applying the hexagon symmetries



$$\left(\prod_{h} \tau_i^z \right)^2 = 1$$

$$\left[\prod_{h} \tau_i^z , \prod_{h'} \tau_j^z \right] = 0 \qquad \forall h, h'$$

Hamiltonian

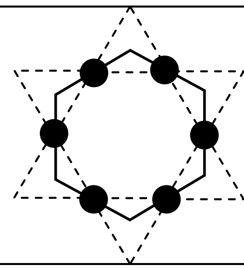
$$H = -\sum_h \underbrace{\prod_{i=1}^{6} \tau_i^z}_{\text{dimer hoping } (=Z_2 \text{ flux})}$$

Solvable kagome dimer model and Kitaev's toric code

Dimer model

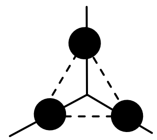
Dimer kinetic energy

$$\prod_{i=1}^{6} \tau_i^z$$



Hard core constraint

$$\tau_1^x \tau_2^x \tau_3^x = 1$$



$$\tau_i^x = \begin{cases} +1 & \text{Arrow with same direction as in the reference config.} \\ -1 & \text{Arrow in opposite direction} \end{cases}$$

Hamiltonian

$$H = -\sum_h \left(\prod_{i=1}^{6} \tau_i^z \right)$$

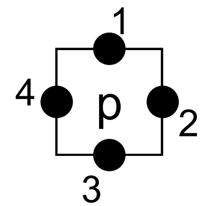
Ground-state

$$|g.s\rangle = \sum_{c \in \{\text{dimer coverings}\}} |c\rangle$$

Kitaev's toric code

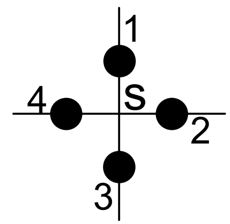
Annals Phys. 303, 2 (2003) [quant-ph/9707021]

Magnetic energy



$$B_p = \sigma_1^z \sigma_2^z \sigma_3^z \sigma_4^z$$

Electric energy



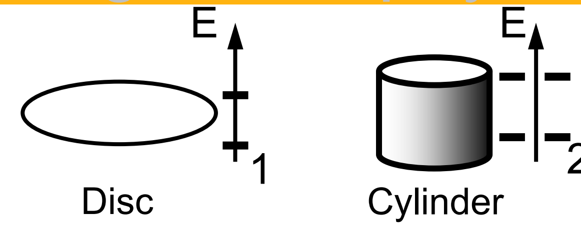
$$A_s = \sigma_1^x \sigma_2^x \sigma_3^x \sigma_4^x$$

$$H = -\sum_s A_s - \sum_p B_p$$

$$|g.s\rangle = \sum_{c \in \left\{ \substack{\text{All } \sigma_i^x = \pm 1 \text{ config.} \\ \text{satisfying } A_s = 1 \ \forall s} \right\}} |c\rangle$$

The simplest Z_2 dimer liquid – short-range RVB phys.

- Topological degeneracy



- No broken symmetry
  - Dimer-dimer correlations are *strictly zero* beyond 2 lattice spacings
  - The degenerate ground-states are locally undistinguishable ⇒ No local order parameter, no (conventional, hidden) order of *any* kind. [Furukawa, GM, Oshikawa PRL 2006]

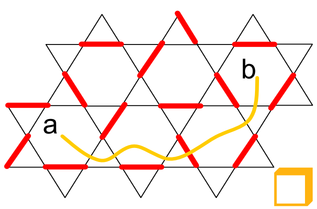
- Gapped excitations

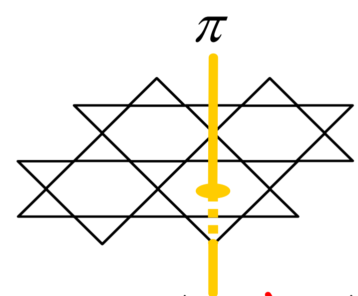
  All wave-functions are known explicitly

  Exact mapping to an Z_2 gauge theory

  Excitations = pairs of (static and non-interacting) Z_2 vortices (visons)

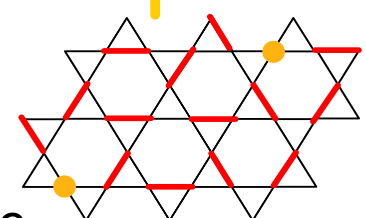
  [Read & Chakraborty PRB 1989; Kivelson PRB 1989; G. X. Wen PRB 1991; Senthil & Fisher PRL 2001]





- Deconfined fractional excitations (monomers)

  doped version of the model: GM, Pasquier, Mila & Lhuillier PRB 2005

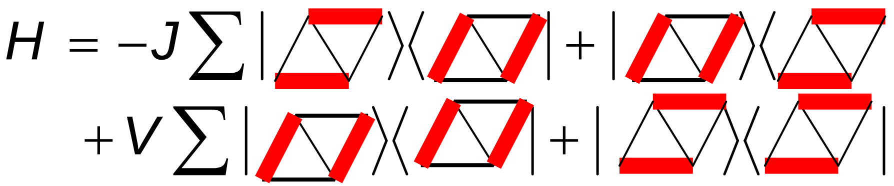


- Perturbation theory ⇒ vortex dispersion relation, PSG, etc.

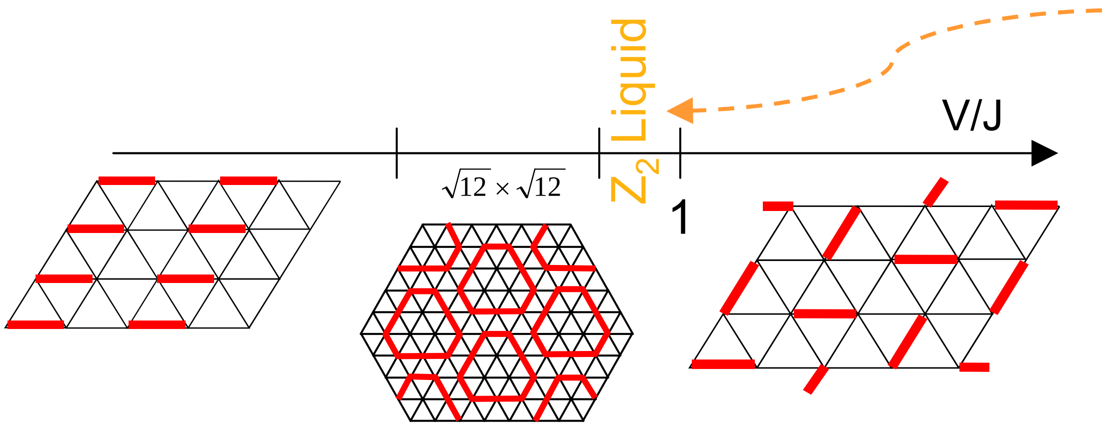
Topological degeneracy in the Z_2 liquid phase of the triangular lattice quantum dimer model

Dimer liquid phase in the triangular QDM

- Quantum dimer model [Moessner & Sondhi, 2001]



- Phase diagram



- Liquid phase:
  - Short-ranged dimer-dimer correlations (but non-zero correlation length)
  - Gapped excitations
  - The ground-state degeneracy depends on the topology.

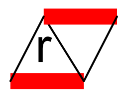
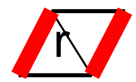
First *simple* model with a short-ranged RVB liquid.
Moessner & Sondhi, Phys. Rev. Lett. (2001)
Ralko *et al.*, Phys. Rev. B 74, 134301 (2006) + refs. therein

Rokhsar-Kivelson point - triangular QDM

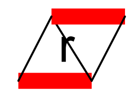
- At J=V=1, the Hamiltonian can be written as

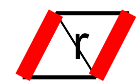
$$H = 2\sum_r |\psi_r\rangle\langle\psi_r|$$

= Sum of projectors

$$|\psi_r\rangle = \frac{1}{\sqrt{2}}\Big(\Big|$$  $$\Big\rangle - \Big|$$  $$\Big\rangle \Big)$$

$$|0\rangle = \sum_c |c\rangle$$

$$\langle\Psi_r|c\rangle = \begin{cases} \frac{1}{\sqrt{2}}|c_{\backslash r\backslash}\rangle \text{ if } |c\rangle = |\cdots \end{cases}$$  $$\cdots\rangle$$

$$-\frac{1}{\sqrt{2}}|c_{\backslash r\backslash}\rangle \text{ if } |c\rangle = |\cdots$$  $$\cdots\rangle$$

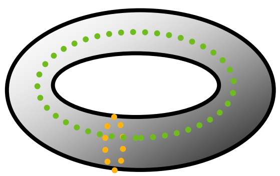
$$0 \text{ otherwise}$$

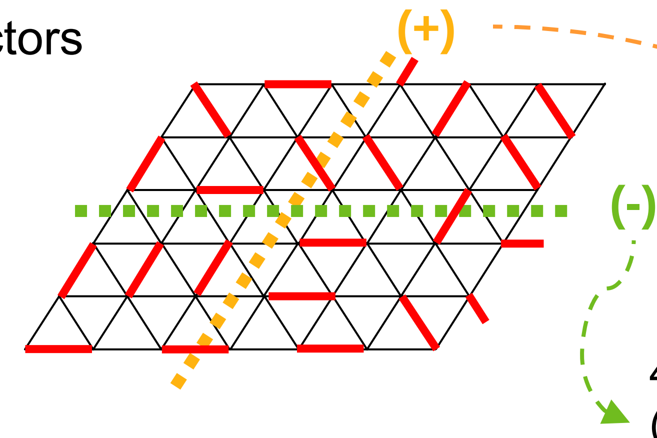
$$\langle\Psi_r|0\rangle = 0$$

$$\Rightarrow H|0\rangle = 0 \Rightarrow \text{a ground-state}$$

Rokhsar-Kivelson point - triangular QDM

- Topological sectors



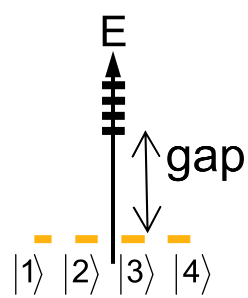


4 sectors defined by the parities (++) (--) (+-) (-+) of the # of dimers crossing the two cuts

$$\begin{cases} |1\rangle = \sum_{c\in(++)} |c\rangle & |2\rangle = \sum_{c\in(+-)} |c\rangle \\ |3\rangle = \sum_{c\in(--)} |c\rangle & |4\rangle = \sum_{c\in(-+)} |c\rangle \end{cases}$$

$$H|i\rangle = 0$$

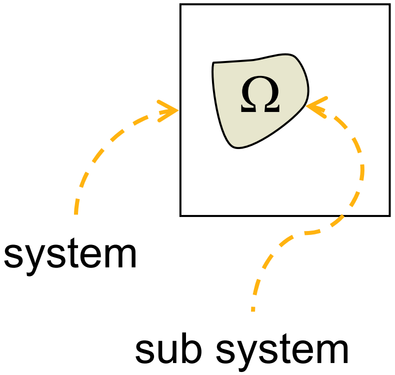
- 4 zero-energy eigenstates (ground-states)
- Finite correlation length
- Excited states not exactly known



- How can we prove that this degeneracy is a due to some topological order, and is not due to some (hidden/complicated) conventional order ?

Measuring the *local* difference between quantum states

Furukawa, GM, Oshikawa PRL 2006



- Measure how much two states are « different » :

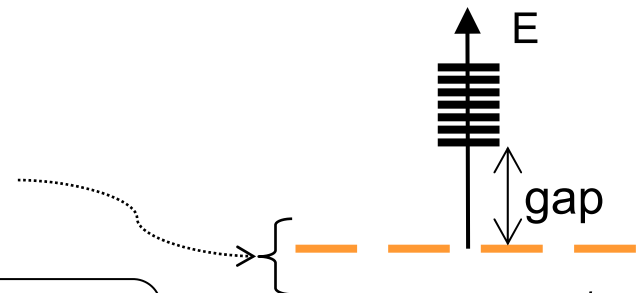
$$\mathrm{diff}\left(|A\rangle,|B\rangle,\Omega\right)=\mathrm{Max}_{\substack{\hat{O}\text{ defined on }\Omega\\ \text{normalized }\|\hat{O}\|\leq 1}}\left|\langle A|\hat{O}|A\rangle-\langle B|\hat{O}|B\rangle\right|$$

$$=\cdots$$

$$=\sum|\lambda_n| \quad \text{where} \quad \underbrace{\rho_A^\Omega-\rho_B^\Omega}_{\text{reduced density matrices}}=\sum\underbrace{\lambda_n}_{\text{eigenvalue}}|n\rangle\langle n|$$

- Generalize to more than two states :

$|1\rangle,|2\rangle,\cdots,|d\rangle$ degenerate ground-states



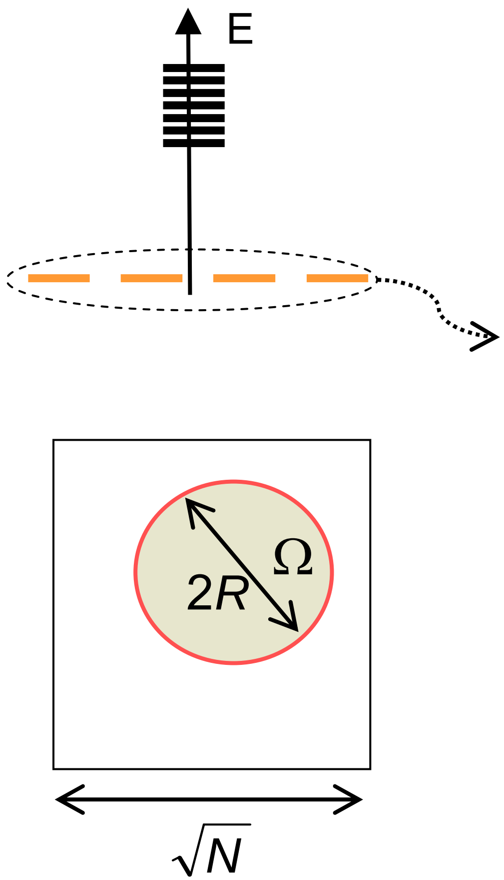
$$D_\Omega\left(|1\rangle,|2\rangle,\cdots,|d\rangle\right)=\mathrm{Max}_{\substack{|A\rangle=\sum\alpha_i|i\rangle\\ \langle A|A\rangle=1}}\mathit{diff}\left(|A\rangle,|\mathit{ref}\rangle,\Omega\right)$$

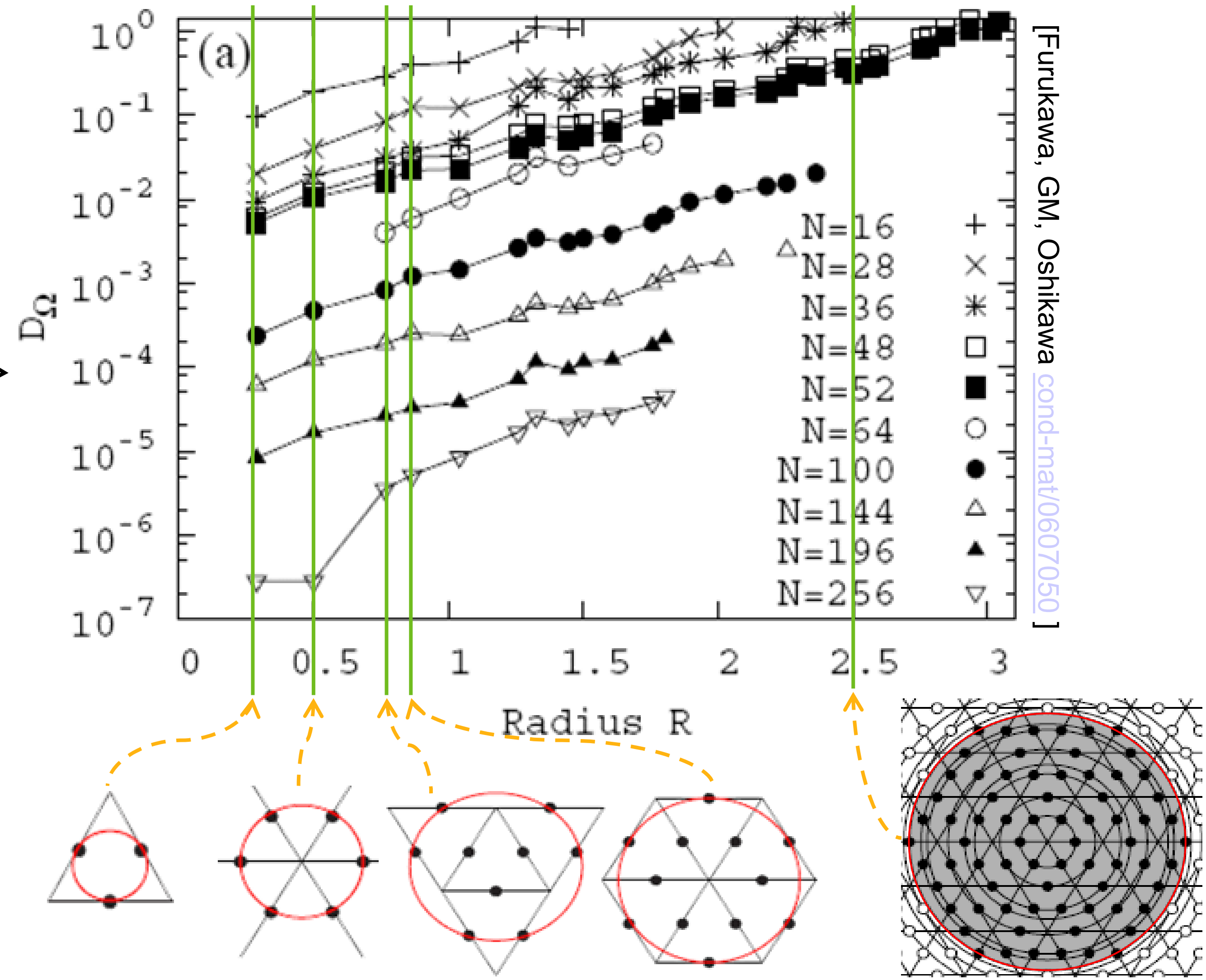
with $|\mathit{ref}\rangle=\frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i\rangle$

- $D_\Omega \to 0$: topological degeneracy
- $D_\Omega = O(1)$: conventional spontaneous symmetry breaking

« Local difference » between the degenerate ground-states

Ground-states of the triangular-lattice quantum dimer model at J=V=1 (RK point)

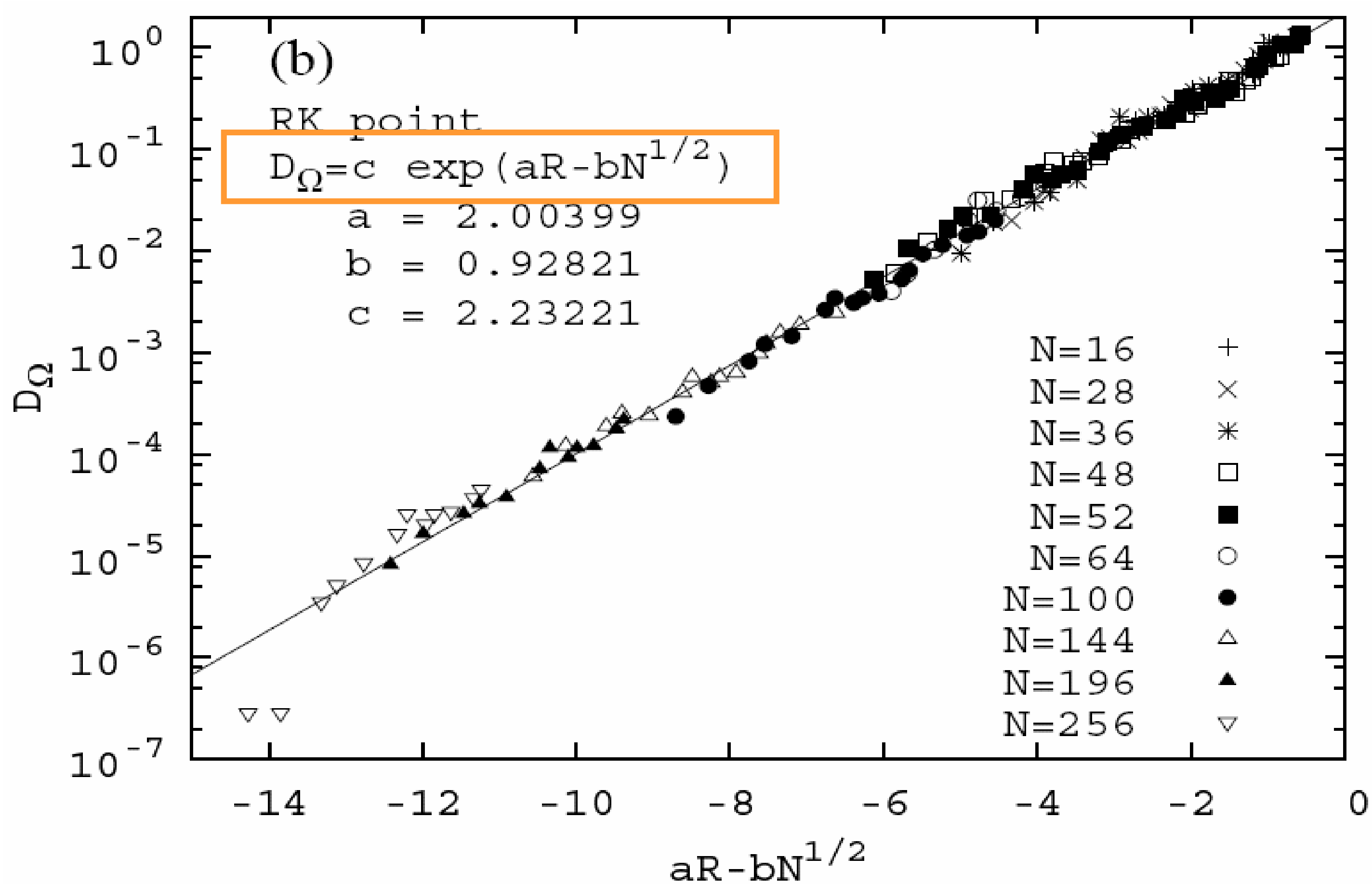




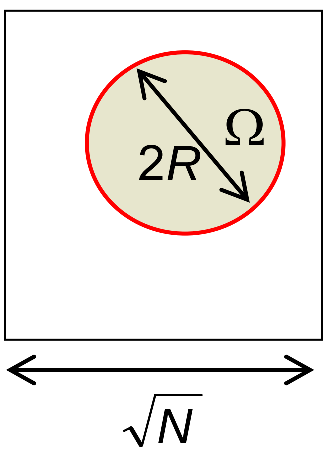
[Furukawa, GM, Oshikawa cond-mat/0607050]

Scaling of the « local difference » D_Ω

Ground-states of the triangular-lattice quantum dimer model at J=V=1 (RK point)



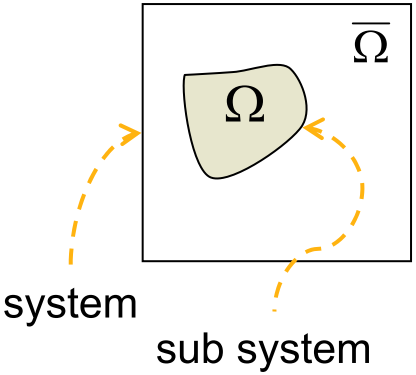
[Furukawa, GM, Oshikawa cond-mat/0607050 (JPMC)]



⇒ The 4 ground-states cannot be distinguished by ***any local observable*** in the thermodynamic limit

Topological entanglement entropy in the liquid phase of the triangular lattice quantum dimer model

Entanglement entropy



Reduced density matrix

$$\rho = \mathrm{Tr}_{\overline{\Omega}}\left[|\psi\rangle\langle\psi| \right]$$

Entanglement entropy

$$S = -\mathrm{Tr}\left[\rho \log \rho\right]$$

Schmidt decomposition

$$|\Psi\rangle = \sum_a \lambda_a |\varphi_a^{\Omega}\rangle |\varphi_a^{\overline{\Omega}}\rangle \quad , \quad \lambda_a > 0$$

$$\langle \varphi_a^{\Omega} | \varphi_b^{\Omega} \rangle = \delta_{ab} \quad , \quad \langle \varphi_a^{\overline{\Omega}} | \varphi_b^{\overline{\Omega}} \rangle = \delta_{ab}$$

$$\rho = \sum_a (\lambda_a)^2 |\varphi_a^{\Omega}\rangle\langle\varphi_a^{\Omega}|$$

$$S = -\sum_a (\lambda_a)^2 \log\left(\lambda_a^{\,2}\right)$$

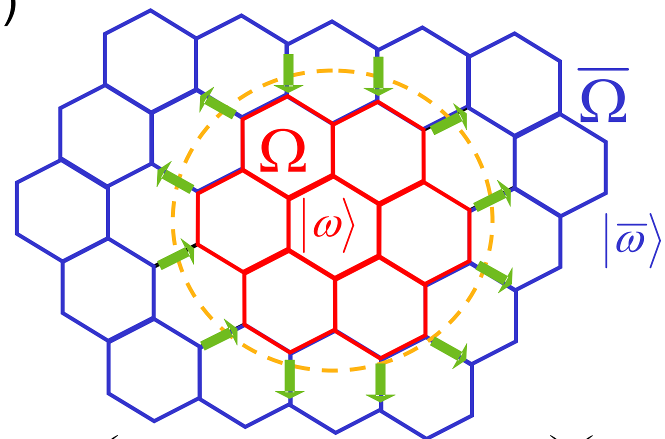
- Idea:

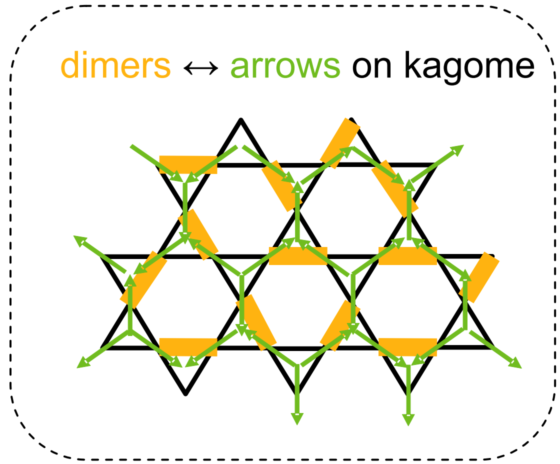
The entanglement entropy of S has contributions
i) coming from local correlations in the vicinity of the boundary,
plus ii) possible non-local contributions.

In a topologically ordered wave-function, this non-local contribution "S_{topo}" may be universal (and equal to the *total quantum dimension*).
See Levin & Wen, and Kitaev & Preskill (PRL 2006).

Entanglement entropy – *kagome* RK wave-function

$|\psi\rangle$ = Equal amplitude superposition of all dimer coverings (on the kagome lattice)





$$|\psi\rangle = \frac{1}{\sqrt{N^{tot}}} \sum_{c \in \{\text{arrows config.}\}} |c\rangle$$

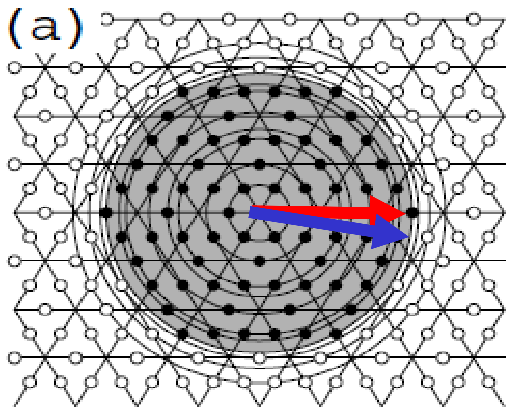
$$= \sum_{b \in \left\{\substack{\text{arrow config.} \\ \text{at the boundary}}\right\}} \underbrace{\frac{\sqrt{N^{\Omega}(b) N^{\overline{\Omega}}(b)}}{\sqrt{N^{tot}}}}_{\text{Schmidt weight}} \left(\frac{1}{\sqrt{N^{\Omega}(b)}} \sum_{\omega \in \left\{\substack{\text{arrow conf.} \\ \text{inside} \\ \text{compat. with } b}\right\}} |\omega\rangle \right) \left(\frac{1}{\sqrt{N^{\overline{\Omega}}(b)}} \sum_{\overline{\omega} \in \left\{\substack{\text{arrow conf.} \\ \text{outside} \\ \text{compat. with } b}\right\}} |\overline{\omega}\rangle \right)$$

$$S = \sum_{b \in \left\{\substack{\text{arrow config.} \\ \text{at the boundary}}\right\}} p_b \log[p_b] \quad , \quad p_b = \frac{N^{\Omega}(b) N^{\overline{\Omega}}(b)}{N^{tot}} = \begin{cases} 2^{1-L} \text{ if } b \text{ has an even \# of incoming arrows} \\ 0 \text{ otherwise} \end{cases}$$

$$S = L \log(2) \qquad \text{Boundary law (local correlations/constraints)}$$
$$-\log(2) \qquad \text{Non-local contribution (global parity constraint)}$$

[Levin & Wen, PRL 2006]

Entanglement entropy - *triangular* RK wave-function

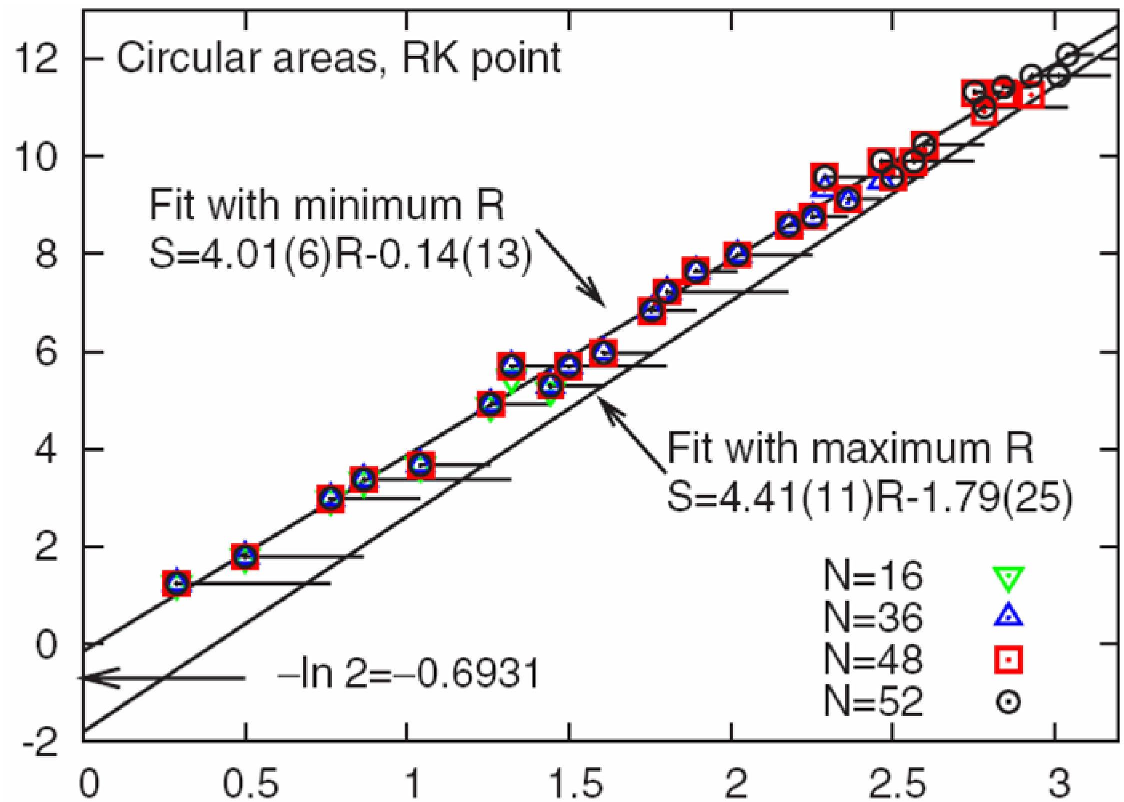


R_{min}=2.50
R_{max}=2.598

We observe a "boundary law", as expected in a gapped system with short-range correlations.

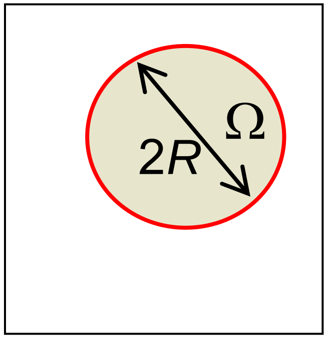
$$S(R) \approx \alpha R + \gamma$$

But ambiguity ~O(1) on R ($R_{min} \neq R_{max}$)
⇒ how to extract γ ?



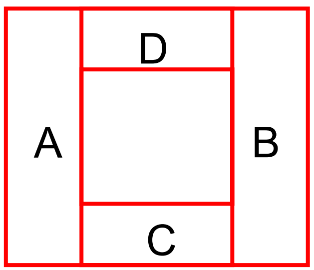
[Furukawa, GM, cond-mat/0612227v2 (Phys. Rev. B)]

Topological entanglement entropy



$$S(R) \approx \alpha R + \gamma$$

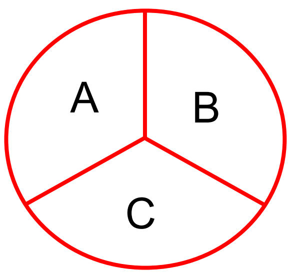
- Two ways to get rid of the local contributions to measure γ :



- M. Levin and X.-G. Wen, Phys. Rev. Lett. 96, 110405(2006)

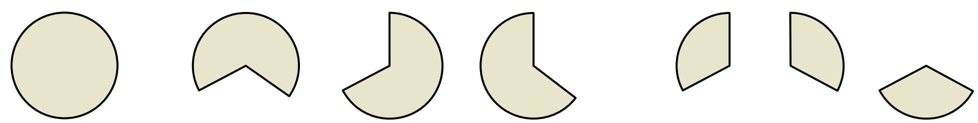
$$2S_{topo} = S_{ABCD} - S_{ABD} - S_{ABC} + S_{AB}$$

=



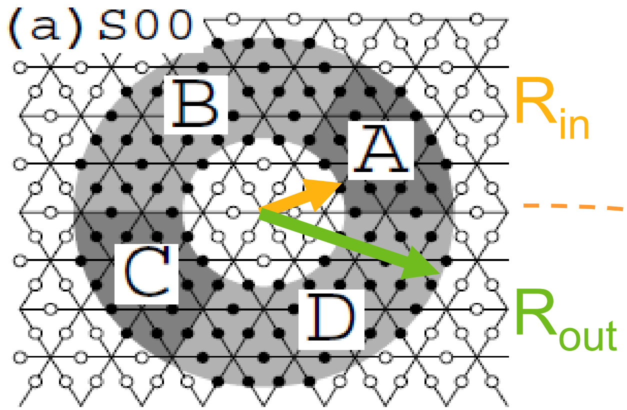
- A. Kitaev and J. Preskill, Phys. Rev. Lett. 96, 110404(2006)

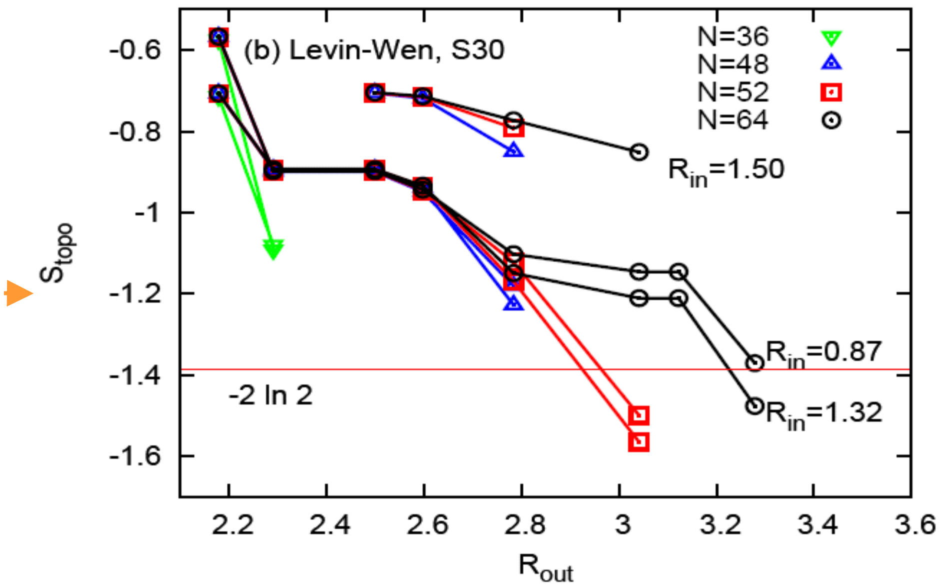
$$S_{topo} = S_{ABC} - \left(S_{AB} + S_{BC} + S_{AC}\right) + \left(S_{A} + S_{B} + S_{C}\right)$$



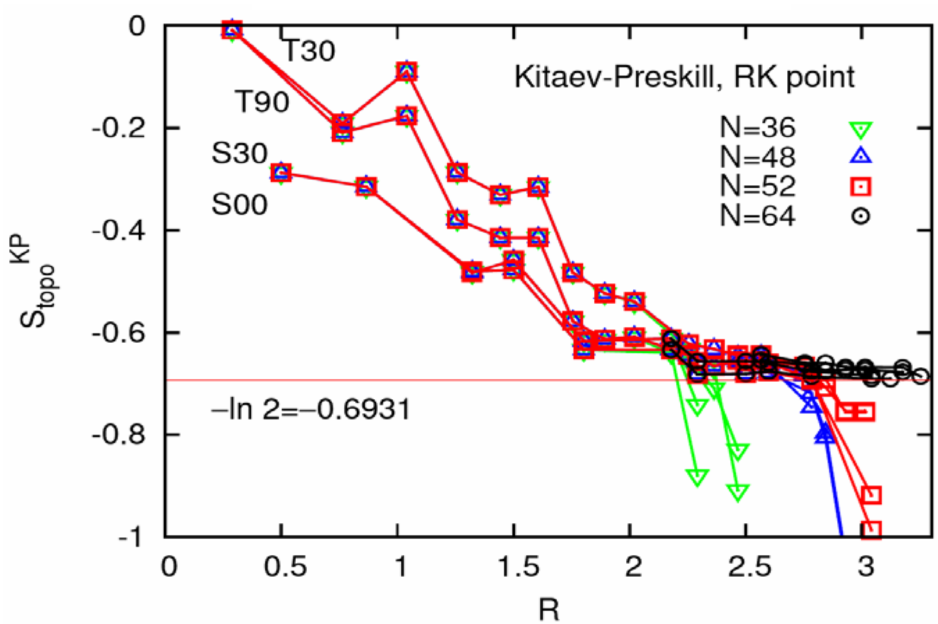
Topological entanglement entropy - *triangular* RK wave-function

- Levin and X.-G. Wen

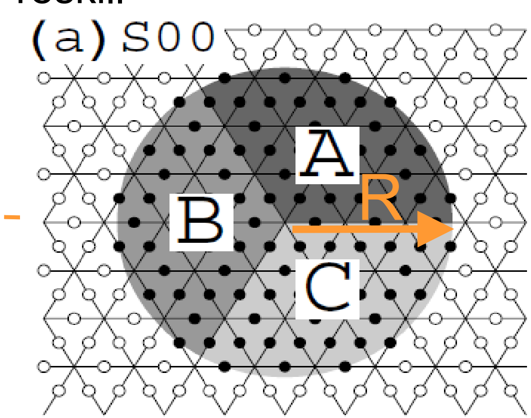




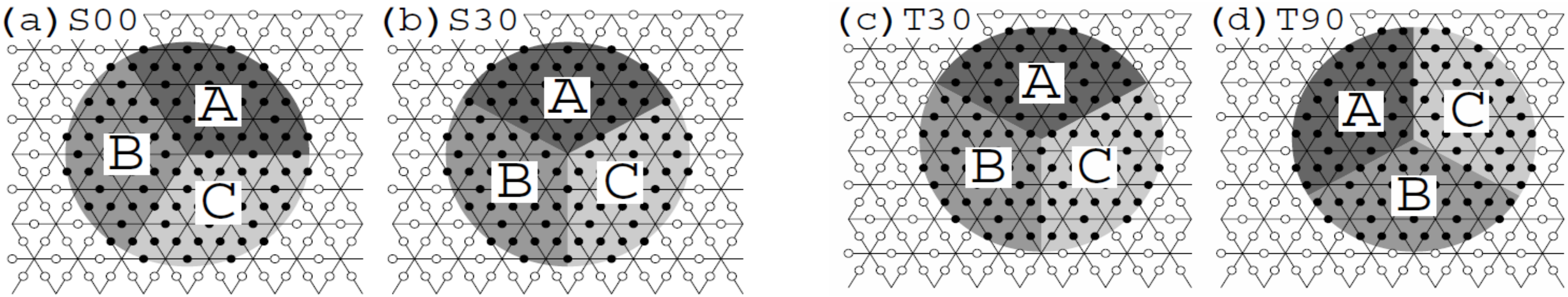
[Furukawa, GM, cond-mat/0612227v2]



- Kitaev and Preskill



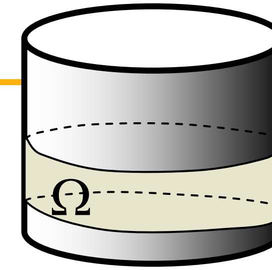
Kitaev-Preskill construction – triangular RK wave-function



	S00 case $-S_{\rm topo}^{\rm KP}/\ln 2$			T30 case $-S_{\rm topo}^{\rm KP}/\ln 2$	
Radius R	$N=52$	$N=64$	Radius R	$N=52$	$N=64$
2.18	0.9143	0.9143	2.57	0.9291	0.9283
2.29	0.9839	0.9835	2.75	0.9618	0.9513
2.50	0.9822	0.9822	2.84	0.9965	0.9518
2.60	0.9765	0.9760	2.93	1.0910	0.9635
2.78	1.0014	0.9897	3.01	1.0910	0.9635
3.04	1.3252	0.9967	3.18		0.9649
3.12		0.9967	3.25		0.9898

[Furukawa, GM, cond-mat/0612227v2 (Phys. Rev. B)]

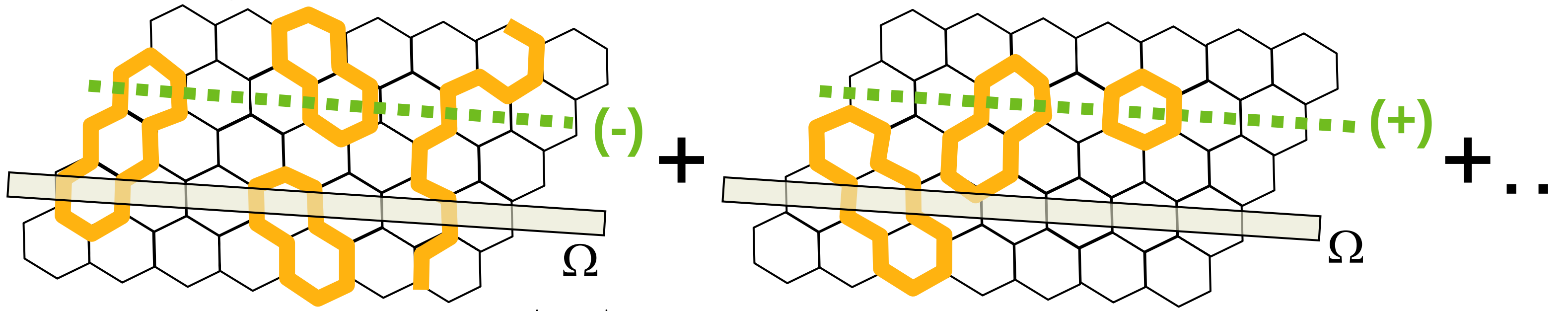
Thin-strip entanglement – kagome RK wave-function



Dimer covering on the kagome lattice ↔ arrow configurations
↔ non-intersecting loops on the honeycomb lattice

- Equal-amplitude ground-state (config. can have an odd or even winding number):

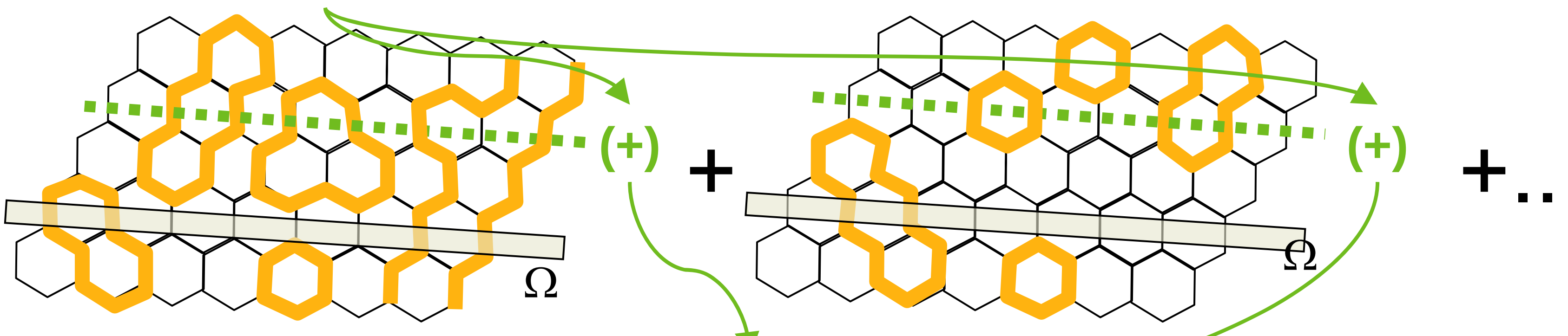
$$\left|RK\right\rangle =$$



$$S_\Omega^{\left|RK\right\rangle} = L_x \log(2)$$

- Fixed-parity-sector (say even) ground-state

$$\left|+\right\rangle =$$

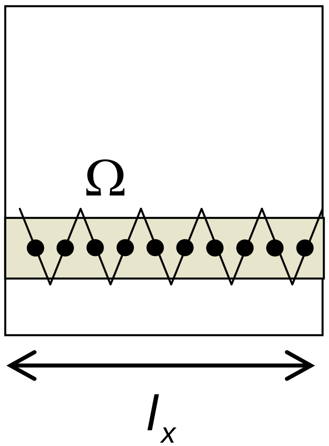


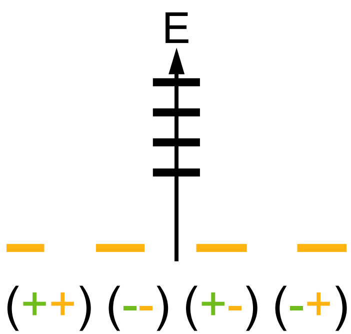
$$S_\Omega^{\left|+\right\rangle} = L_x \log(2) \boxed{-\log(2)}$$

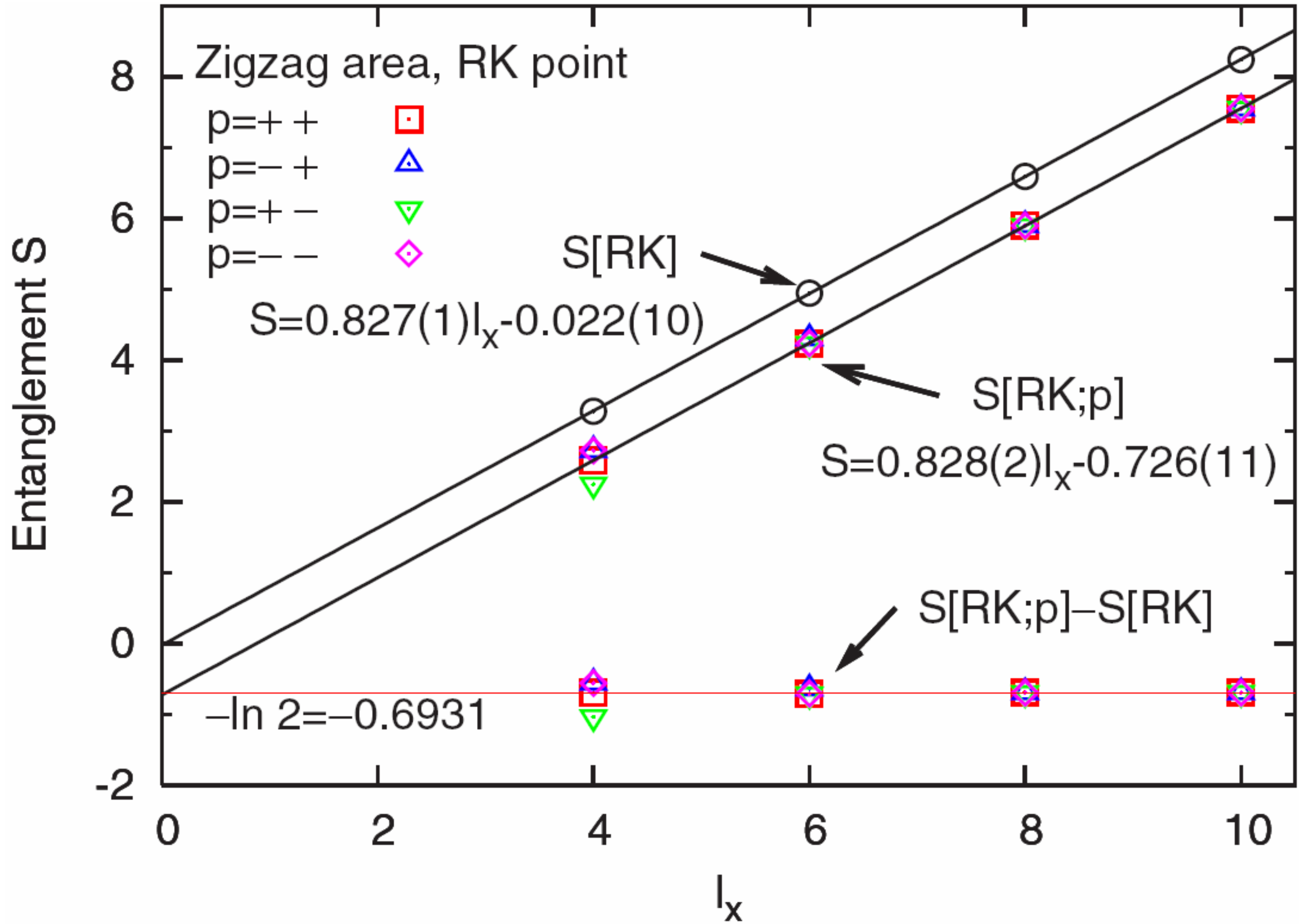
[Hamma *et al.* PRA 2005]

Topological entanglement entropy - *triangular* RK wave-function

- Use a « thin strip » winding around the system ⇒ obtain γ=-log(2) with 0.2 % accuracy.

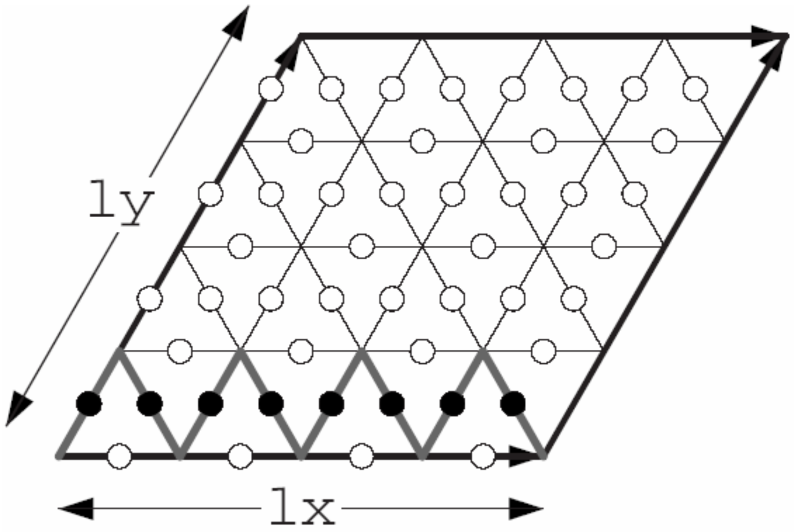






[Furukawa, GM, cond-mat/0612227v2]

Thin-strip construction – triangular RK wave-function

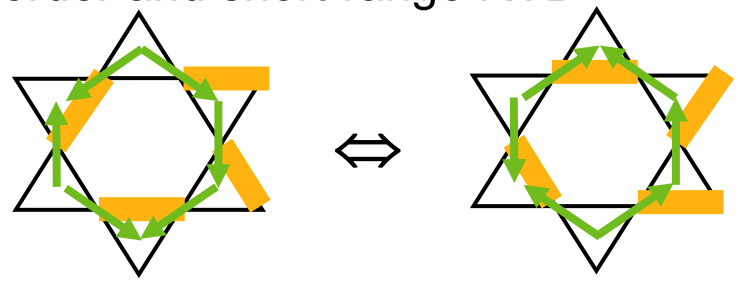


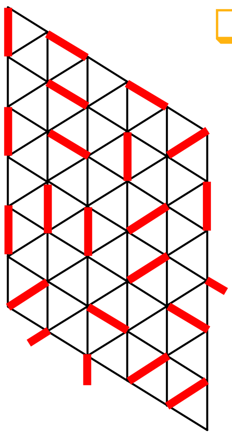
$l_x(=l_y)$	$(S[\mathrm{RK}]-S[\mathrm{RK};p])/\ln 2$ ++	−+	+−	−−
4	1.0024*	0.8051	1.4910	0.8051
6	1.0315	0.9248	1.0315	1.0212*
8	0.9944*	1.0022	1.0017	1.0022
10	0.9981	1.0028	0.9981	1.0011*

[Furukawa, GM, cond-mat/0612227v2 (Phys. Rev. B)]

Summary

- Quantum dimer model version of Kitaev toric code :
  - solvable toy model
  - explicit connection between Z_2 topological order and short-range RVB physics





- Triangular-lattice quantum dimer model:
  - Numerical investigation of the “undistinguishable character” of the different ground-states.
  - One of the first numerical investigation of the “topological entanglement entropy” introduced by Levin-Wen & Kitaev-Preskill (2006).
  - Thin strip construction. Confirmation of γ=-log(2).