From Amplitudes to Wilson Loops

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based on work done in collaboration with:

Heslop & Travaglini 0707.1153
Heslop, Nasti, Spence & Travaglini 0805.2763

Outline

• Motivation
• MHV amplitudes in N=4 SYM & Wilson loops at weak coupling
• Iterative structures of loop amplitudes in N=8 SUGRA and Wilson loops
• Conclusions
Motivation

- Why are amplitudes so simple and how can we make use of this observation?
  - Geometry in twistor space (Witten 2003)
  - Iterative structures in S-matrix of gauge theory & gravity
- Simplicity hidden by standard Feynman rules
  - no manifest gauge symmetry
  - unphysical poles
- (Generalised) Unitarity & Twistor inspired methods
  - only gauge invariant, on-shell quantities enter at intermediate steps
  - apply also in non-supersymmetric theories
• In theories with maximal supersymmetry amplitudes are particularly simple ⇒ Ideal laboratory to test new ideas

• N=4 SYM: colour ordered & planar limit (leading in $1/N$)

• All one-loop amplitudes are linear combination of box functions (Bern-Dixon-Dunbar-Kosower), coefficients from generalised unitarity (Britto-Cachazo-Feng)

• Recursive structures in higher loop splitting amplitudes and MHV amplitudes (Anastasiou-Bern-Dixon-Kosower, Bern-Dixon-Smirnov)

• Splitting amplitudes: universal, govern collinear limits

• MHV: gluon helicities are permutation of $-+-+-++$
• Surprising relation to lightlike Wilson loops:
  strong coupling: (Alday-Maldacena)  Alday’s talk
  weak coupling: (Drummond-Korchemsky-Sokatchev+Henn, AB-Heslop-Travaglini)

• Dual conformal symmetry

• integral functions in planar amplitudes
  (Drummond-Henn-Smirnov-Sokatchev)

• Wilson loops (Alday-Maldacena, Drummond-Henn-Korchemsky-Sokatchev)

• Maximal transcendentality
Today, consider **MHV amplitudes in N=4 SYM and N=8 SUGRA**

- Some common features of N=4 & N=8
  - Tree level recursion relations, good UV behaviour under complex shifts (Bedford, AB, Spence, Travaglini; Cachazo-Svrcek; Benincasa, Boucher-Veronneau, Cachazo; Arkani-Hamed, Kaplan; Bianchi-Elvang-Freedman)
  - One-loop: “No Triangle Hypothesis” (Bern, Dixon, Perelstein, Rozowsky; Bern, Bjerrum-Bohr, Dunbar; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager; Bjerrum-Bohr, Vanhove)
  - Both are important for possible finiteness of N=8 SUGRA (Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager; Bern, Dixon, Roiban; Green, Russo, Vanhove; Bern, Carrasco, Dixon, Johansson, Kosower, Roiban)

- Transcendentality
Goals for the rest of the talk

- MHV amplitudes in N=4 SYM
- Iterative structures in perturbative expansion (Korchemsky’s talk)
- Relate one-loop $n$-gluon amplitudes to Wilson loops (AB-Heslop-Travaglini)
- 4-graviton MHV amplitude in N=8 SUGRA
  - Look for iterative structures (similar to N=4)
  - Try to find relation to Wilson loops
N=4 SYM

- Simplest one-loop amplitude is the n-point MHV amplitude in N=4 SYM at one loop (colour-ordered, partial amplitude):

\[ A_{1-loop}^{MHV} = A_{tree}^{MHV} \sum_{p,q} 1 \times \]

- Calculated using unitarity in 1994 (Bern-Dixon-Dunbar-Kosower)
- Rederived from MHV diagrams in 2004 (AB-Spence-Travaglini)
- From Wilson loop in 2007 (AB-Heslop-Travaglini)
Suprising iterative structure at two loops...

- **n-point MHV amplitude in N=4:** \( A_n^{(L)} = A_n^{tree} M_n^{(L)} \)

- **First observed for 4 gluon scattering in planar N=4 SYM at 2 loops** (Anastasiou-Bern-Dixon-Kosower)

\[
M_n^{(2)}(\epsilon) - \frac{1}{2} \left( M_n^{(1)}(\epsilon) \right)^2 = f^{(2)}(\epsilon) M_n^{(1)}(2\epsilon) + C^{(2)} + O(\epsilon)
\]

contains anomalous dimension of twist two operators at large spin

- **Requires knowledge of one-loop amplitude to higher, positive orders in** \( \epsilon \), \( D = 4 - 2\epsilon \), in dimensional regularisaiton
...and even higher loops

In 2005 Bern-Dixon-Smirnov (BDS) found a similar iterative structure for $n=4$ at 3 loops and proposed an all-loop order formula for the MHV amplitudes in planar N=4 SYM.

\[ a \sim g_{YM}^2 N/(8\pi^2) \]

\[ \mathcal{M}_n := 1 + \sum_{L=1}^{\infty} a^L \mathcal{M}^{(L)}_n = \exp \left[ \sum_{L=1}^{\infty} a^L \left( f^{(L)}(\epsilon) \mathcal{M}^{(1)}_n(2\epsilon) + C^{(L)} + O(\epsilon) \right) \right] \]

- $\mathcal{M}^{(1)}_n$ is the all orders in $\epsilon$ one-loop MHV amplitude
- In order to extract recursive relations order-by-order in $a$ consider the log of this expression, e.g. for $L=2$ & 3

\[ \mathcal{M}^{(2)}_n = \frac{1}{2} \left( \mathcal{M}^{(1)}_n(\epsilon) \right)^2 + f^{(2)}(\epsilon) \mathcal{M}^{(1)}_n(2\epsilon) + O(\epsilon) \]

\[ \mathcal{M}^{(3)}_n = -\frac{1}{3} \left( \mathcal{M}^{(1)}_n(\epsilon) \right)^3 + \mathcal{M}^{(2)}_n(\epsilon) \mathcal{M}^{(1)}_n(\epsilon) + f^{(3)}(\epsilon) \mathcal{M}^{(1)}_n(3\epsilon) + O(\epsilon) \]
Comments

• The exponential form is strongly motivated by the universal factorisation & exponentiation/resummation of IR divergences in gauge theories (not only N=4)

• The miracle in N=4 is that exponentiation also applies to the finite parts of the amplitude and the finite remainder becomes a constant independent of kinematics

• Confirmed by a recent strong coupling calculation using AdS/CFT by Alday-Maldacena (at least for n=4).
Test of the conjecture

- **Two and three loops, n=4** (Anastasiou, Bern, Dixon, Kosower; Bern, Dixon, Smirnov)
- **Two loops, n=5** (Bern, Czakon, Kosower, Roiban, Smirnov; Cachazo, Spradlin, Volovich)
- **Problems for more gluons**
  - $n \rightarrow \infty$ (Alday, Maldacena)
- **n=6** (Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich; Cachazo, Spradlin, Volovich)
- **Exponent requires an additional kinematic dependent finite remainder function**
Amplitudes and Wilson Loops

(Drummond, Korchemsky, Sokatchev; AB, Heslop, Spence, Travaglini)

- MHV amplitudes in N=4 super Yang-Mills \( \mathcal{A}_n^{(L)} = \mathcal{A}_{n_{\text{tree}}}^{\text{tree}} \mathcal{M}_n^{(L)} \)
- Surprisingly, \( \mathcal{M}_n \) appears in a very different context!

**Wilson loop calculation**

- **Lightlike Contour** in dual momentum space \( \Rightarrow \) fixed by on-shell momenta of gluons (colour-ordered, helicity-blind)

\[
p_i = k_i - k_{i+1} \quad \text{\( k \)'s are (T-)dual momenta}
\]

\[
\sum_{i=1}^{n} p_i = 0 \quad \Rightarrow \text{Contour is closed}
\]
Amplitudes and Wilson Loops

- Calculate \( < W[C] > \) at weak coupling

\[
W[C] := \text{Tr} P \exp \left[ ig \oint_C d\tau \left( A_\mu(x(\tau)) \dot{x}^\mu(\tau) + \phi_i(x(\tau)) \dot{y}^i(\tau) \right) \right]
\]

- Contour \( C \) of previous page is the same as in the strong coupling calculation of Alday-Maldacena using AdS/CFT

- When \( \dot{x}^2 = \dot{y}^2 \) Wilson loop is locally supersymmetric

- Here we have \( \dot{x}^2 = 0 \) (lightlike momenta) and \( \dot{y} = 0 \)

- Locally Supersymmetric
Motivation: recent computation of **gluon amplitudes at strong coupling** (Alday-Maldacena)

- scattering in AdS is at **fixed angle, high energy** $\Rightarrow$ similar to Gross-Mende calculation
- $\Rightarrow$ exponential of classical string action $A \sim e^{-S_{\text{cl}}} = e^{-\sqrt{\lambda}/(2\pi)(\text{Area})_{\text{cl}}}$
- In T-dual variables the B.C.s of the string is a **lightlike polygonal loop** $C$ embedded in the boundary of AdS
- Finding the **minimal area** with these B.C.s is equivalent to the calculation of a **lightlike Wilson loop** in AdS/CFT (Maldacena; Rey-Yee)
- **Alday-Maldacena**: confirmation of BDS conjecture at 4-points at strong coupling!
< $W[C]$ > and MHV amplitudes at 1-loop

(AB, Heslop, Travaglini)

- **Two classes of diagrams** *(Feynman gauge)*:

  A. **IR divergent**

  B. **Finite**

Gluon stretched between two segments **meeting at a cusp**

Gluon stretched between two **non-adjacent** segments
• Clean separation of IR divergent and Finite terms

• From diagrams in class A: \[ \mathcal{M}_n^{(1)}|_{IR} = -\frac{1}{\epsilon^2} \sum_{i=1}^{n} \left( \frac{-s_{i,i+1}}{\mu^2} \right)^{-\epsilon} \]

• \( s_{i,i+1} = (p_i + p_{i+1})^2 \) is the invariant formed from the momenta meeting at the cusp

• Diagrams in class B give rise to the following integral

\[
\mathcal{F}_\epsilon(s,t,P,Q) = \int_0^1 d\tau_p d\tau_q \frac{P^2 + Q^2 - s - t}{\left[-(P^2 + (s - P^2)\tau_p + (t - P^2)\tau_q + (-s - t + P^2 + Q^2)\tau_p\tau_q)\right]^{1+\epsilon}}
\]

• equal to the finite part of 2-mass easy box function!

• Comment: this integral is directly related to the Feynman parameter integral of the 2-mass easy box function
• In the example: \( p = p_2 \quad q = p_5 \)

\[
P = p_3 + p_4, \quad Q = p_6 + p_7 + p_1
\]

• One-to-one correspondence between Wilson loop diagrams and finite parts of 2-mass easy box functions

• “Explains” why box functions appear with coefficient = 1 in the one-loop N=4 MHV amplitude
Explicit calculation gives at $\varepsilon \to 0$:

$$a := \frac{2(pq)}{P^2Q^2 - st}$$

$$\mathcal{F}_{\varepsilon=0} = -\text{Li}_2(1-as) - \text{Li}_2(1-at) + \text{Li}_2(1-aP^2) + \text{Li}_2(1-aQ^2)$$

finite part of the box function appearing in the one-loop MHV amplitude in $\mathcal{N}=4$ SYM!

At 4 points, find the correct all-orders in $\varepsilon$ result (terms up to $O(\varepsilon)$ agree with result of Drummond-Korchemsky-Sokatchev):

$$\mathcal{M}^{(1)}_4(\varepsilon) = -\frac{2}{\varepsilon^2} \left[ \left( \frac{-s}{\mu^2} \right)^{-\varepsilon} 2F_1 \left( 1, -\varepsilon, 1 - \varepsilon, 1 + \frac{s}{t} \right) + \left( \frac{-t}{\mu^2} \right)^{-\varepsilon} 2F_1 \left( 1, -\varepsilon, 1 - \varepsilon, 1 + \frac{t}{s} \right) \right]$$

For $n > 4$, missing topologies (vanish as $\varepsilon \to 0$)

E.g. for $n=5$ amplitude contains parity odd term (pentagon integral). Wilson loop does not capture that!
Comment: “conformal gauge”

- Consider gluon propagator \( \sim \left( 1 + \frac{1}{\epsilon} \right) \frac{\eta_{\mu\nu} - 2x_\mu x_\nu/x^2}{(-x^2 + i\epsilon)^{1-\epsilon}} \)

- **Type A** diagrams vanish!

- **Type B** diagrams are in one-to-one correspondence with complete 2-mass easy box functions incl. IR-div. terms

- would be interesting to investigate this further for **higher loops**

- We will come back to a similar gauge later when we consider Wilson loops for **amplitudes in N=8 SUGRA**
Key result: non-abelian exponentiation theorem
(Gatheral; Frenkel-Taylor)

\[ \langle W[C] \rangle := 1 + \sum_{L=1}^{\infty} a^L W^{(L)} = \exp \sum_{L=1}^{\infty} a^L w^{(L)} \]

\(w's\) are calculated by keeping only the subset of diagrams containing maximal non-abelian colour factor

Also: exponential form of the answer is automatic
Two conjectures

- the Wilson loop and BDS conjecture can be written as

\[ M_n = 1 + \sum_{L=1}^{\infty} M_n^{(L)}(\epsilon) = \exp \left( \sum_{L=1}^{\infty} m_n^{(L)} + O(\epsilon) \right) \]

- it’s more illuminating to write the log of this; expanding to e.g. 3 loop order ⇒

\[ M_n^{(1)} = m_n^{(1)} + O(\epsilon) \]
\[ M_n^{(2)} - \frac{1}{2} \left( M_n^{(1)} \right)^2 = m_n^{(2)} + O(\epsilon) \]
\[ M_n^{(3)} + \frac{1}{3} \left( M_n^{(1)} \right)^3 - M_n^{(1)} M_n^{(2)} = m_n^{(3)} + O(\epsilon) \]

Note: RHS is parity even, hence the parity odd terms on LHS must cancel to order \( O(\epsilon) \)!
Checks of the conjectures

- BDS conjecture: \[ m_n^{(L)} = a^L [f^{(L)}(\epsilon) M^{(1)}(L\epsilon) + C^{(L)}] + O(\epsilon) \]
- Wilson loop conj.: \[ m_n^{(L)} = a^L w_n^{(L)} + O(\epsilon) \]

- Checks of BDS conjecture:
  - ✓ n=4 up to L=3 (BDS)
  - ✓ n=5 up to L=2 (Cachazo-Spradlin-Volovich, Bern-Czakon-Kosower-Roiban-Smirnov)
  - ★ Problems starting at n=6 at L=2, finite remainder (Bern-Dixon-Kosower-Roiban-Spradlin-Vergu-Volovich, Cachazo-Spradlin-Volovich)

- Checks of Wilson loop conjecture:
  - ✓ all n at L=1 (Drummond-Korchemsky-Sokatchev, AB-Heslop-Travaglini)
  - ✓ n=4, 5, 6 at L=2 (Drummond-Henn-Korchemsky-Sokatchev)
We wish to address 2 questions:

- Do amplitudes in \( \text{N}=8 \) SUGRA exhibit iterative structures as in \( \text{N}=4 \) SYM? (Naculich-Nastase-Schnitzer, AB-Heslop-Nasti-Spence-Travaglini)

- Is there an analogous Wilson loop/Amplitude duality for MHV amplitudes? (AB-Heslop-Nasti-Spence-Travaglini)

Focus on four graviton amplitudes

- Tree-level amplitude factors out

\[
\mathcal{A}_4^{(L)} = \mathcal{A}_4^{\text{tree}} \mathcal{M}_4^{(L)}
\]
Looking for iterative structures

- As in N=4 SYM write:

\[ \mathcal{M}_4 = 1 + \sum_{L=1}^{\infty} \mathcal{M}^{(L)}_4 = \exp \left[ \sum_{L=1}^{\infty} m^{(L)}_4 \right] \]

\[ m^{(1)}_4 = \mathcal{M}^{(1)}_4, \quad m^{(2)}_4 = \mathcal{M}^{(2)}_4 - \frac{1}{2} (\mathcal{M}^{(1)}_4)^2 \]

- We want to find \( m^{(2)}_4 \)

- **Recall:** in N=4 SYM this term was proportional to the one-loop amplitude for 4 and 5 gluons and hence IR divergent.
Amplitudes in N=8 SUGRA

• Tree-level:
  • KLT (Kawai-Lewellen-Tye)
  • On-shell Recursions (Bedford-AB-Spence-Travaglini, Cachazo-Svrcek). Suprisingly good UV behaviour under complex shifts

• One-loop: sum of box functions ⇨ “no-triangle hypothesis”
  • MHV amplitudes: 4 point (Green-Schwarz-Brink, Dunbar-Norridge); general case from unitarity (Bern-Dixon-Perelstein-Rozowsky). MHV-Amplitude = ⟨ij⟩^8 x (helicity blind function)
  • non-MHV amplitudes: many examples from generalised unitarity (Bern, Bjerrum-Bohr, Dunbar, Ita)

• 2-loop, 4 point (Bern-Dunbar-Dixon-Perelstein-Rozowsky)
  3-loop, 4 point (Bern-Carrasco-Dixon-Johansson-Kosower-Roiban)
IR divergences

• **One-loop IR divergences** known to **exponentiate**, similar to QED. Weinberg’s proof used **eikonal approximation**

• **IR behaviour** is **softer** compared to YM. At one loop only \( \frac{1}{\epsilon} \)

• E.g. for 4 points at one loop (Dunbar, Norridge)

\[
\mathcal{M}^{(1)}\bigg|_{\text{IR}} = c_T \left( \frac{\kappa}{2} \right)^2 \frac{2}{\epsilon} \left( s \log(-s) + t \log(-t) + u \log(-u) \right)
\]

• **Absence of colour ordering**

\[
\mathcal{M}\bigg|_{\text{IR}} = \prod_{i<j} \mathcal{M}_{\text{div}}(s_{ij})
\]

• Also, soft and collinear amplitudes tree level exact (Bern, Dunbar, Dixon, Perelstein, Rozowsky)
One- and two-loop 4-point amplitudes

- **One-loop** (Green-Schwarz-Brink, Dunbar-Norridge)
  - no colour ordering $\Rightarrow$ answer involves sum over permutations $(1234), (1423), (1342)$

$$
\mathcal{M}^{(1)}_4 = -i st u \left(\frac{\kappa}{2}\right)^2 \left[ \mathcal{I}^{(1)}_4(s, t) + \mathcal{I}^{(1)}_4(s, u) + \mathcal{I}^{(1)}_4(u, t) \right]
$$

$$
\mathcal{I}^{(1)}_4(s, t) := \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2(l - p_1)^2(l - p_1 - p_2)^2(l + p_4)^2}
$$

zero-mass box function
Two-loop: (Bern-Dunbar-Dixon-Perelstein-Rozowsky)

\[ M_{4}^{(2)} = \left( \frac{\kappa}{2} \right)^{4} stu \left[ s^{2} I_{4}^{(2),P}(s,t) + s^{2} I_{4}^{(2),P}(s,u) + s^{2} I_{4}^{(2),NP}(s,t) + s^{2} I_{4}^{(2),NP}(s,u) + \text{cyclic} \right] \]

Where \( I_{4}^{(2),P}, I_{4}^{(2),NP} \) are the planar and non-planar double boxes

Calculated analytically in DR by Smirnov and Tausk

Note: the non-planar integral is not transcendental

Starting point to study possible iterations
Main result: \[ M_4^{(2)} - \frac{1}{2} \left( M_4^{(1)} \right)^2 = \text{finite} + O(\epsilon) \]

Finite remainder has uniform transcendentality

Specific combination of NP boxes is transcendental

Does this persist to higher loops?

Remainder is not related to one-loop amplitude (unlike 4 point N=4 SYM amplitude) and contains logarithms and (Nielsen) polylogs.

Answer is in agreement with the expected exponentiation of the one loop IR divergences, i.e. the remainder function is finite
• the full answer is

\[ M_4^{(2)} - \frac{1}{2}(M_4^{(1)})^2 = -\left(\frac{\kappa}{8\pi}\right)^4 \left[u^2[k(y) + k(1/y)] + s^2[k(1 - y) + k(1/(1 - y)]
\right.
\]
\[ +t^2[k(y/(y - 1)) + k(1 - 1/y)]] + O(\epsilon) \]

where

\[
k(y) := \frac{L^4}{6} + \frac{\pi^2 L^2}{2} - 4S_{1,2}(y)L + \frac{1}{6}\log^4(1 - y) + 4S_{2,2}(y) - \frac{19\pi^4}{90}
\]
\[ +i \left( -\frac{2}{3}\pi \log^3(1 - y) - \frac{4}{3}\pi^3 \log(1 - y) - 4L\pi \text{Li}_2(y) + 4\pi\text{Li}_3(y) - 4\pi\zeta(3) \right) \]

and \( y = -s/t, \ L := \log(s/t) \)
Wilson loops for gravity amplitudes

- Properties of candidate Wilson loop:
  - contour fixed by momenta of gravitons
  - invariant under diffeos
  - same symmetries as scattering amplitude
- As in eikonal approximation we do not expect to capture the helicity dependence
Holonomy

• Natural starting point would be the holonomy of the Christoffel connection $\Gamma$, $\langle \text{Tr} U(C) \rangle$ with

\[ U^\alpha_\beta(C) := \mathcal{P} \exp \left[ i\kappa \oint_C dy^\mu \Gamma^\alpha_{\mu\beta}(y) \right] \]

• Studied by Modanese in perturbation theory
  • Invariant under diffeos ...
  • ... but answer has nothing to do with an amplitude.

\[ \kappa^2 \int_C dx^\mu dy^\nu \langle \Gamma^\alpha_{\mu\beta}(x) \Gamma^\beta_{\nu\alpha}(y) \rangle \sim \kappa^2 \int_C dx_\mu dy^\mu \delta^{(D)}(x - y) \]
Eikonal Wilson loop

• Try an expression that has been used in the past for calculations of amplitudes involving gravitons in the eikonal approximation (Kabat-Ortin, Fabbrichesi-Pettorino-Veneziano-Vilkovisky)

• In linearised approximation

\[ g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x) \]

\[ W[C] := \langle \mathcal{P} \exp \left[ i\kappa \oint_C d\tau \ h_{\mu\nu}(x(\tau)) \dot{x}^\mu(\tau) \dot{x}^\nu(\tau) \right] \rangle \]

• The exponent can be written as

\[ \int d^Dx T^{\mu\nu}(x) h_{\mu\nu}(x) \]

where the EM-tensor is that of a free point particle

• However, if the contour \( C \) has cusps, then the loop is not diffeomorphism invariant!
• Try anyway!

• First, in order to implement the symmetries of the amplitude we propose to consider

\[ W := W[C_{1234}] W[C_{1423}] W[C_{1342}] \]

• At one loop this becomes

\[ W^{(1)} := W^{(1)}[C_{1234}] + W^{(1)}[C_{1423}] + W^{(1)}[C_{1342}] \]

where the contours \( C_{ijkl} \) are constructed by connecting the momenta in the prescribed order
There are two classes of diagrams as in N=4 SYM. (A sum over cyclic permutations in (234) is understood)

- **A. IR divergent**
- **B. Finite**
From diagrams in **Class A** we get:

$$\kappa^2 \frac{c(\epsilon)}{\epsilon^2} \left[ (-s)^{1+\epsilon} + (-t)^{1+\epsilon} + (-u)^{1+\epsilon} \right]$$

- The leading divergence cancels since \( s + t + u = 0 \).
- Subleading terms as expected

From diagrams in **Class B** we get:

$$c(\epsilon) \frac{u}{2} \frac{1}{4} \left[ \log^2 \left( \frac{s}{t} \right) + \pi^2 \right]$$

- This is the finite part of a zero mass box function. Sum over perms reproduces the finite part of amplitude
Summary of Results

- **Tree level factor** missing (as in N=4 SYM)

- **Relative normalisation** between IR divergent and finite terms is incorrect by a factor of (-2)
  
  - A factor of 2 can be accounted for by an effective overcounting of cusp contributions in \( W \); the **minus sign** is harder to explain

- The result is **gauge dependent** (so far we were using de Donder gauge), **but close to the correct answer**...
Conformal Gauge

- Defined as the gauge where the cusp diagrams vanish
- have illustrated that earlier for Yang-Mills, where Wilson loop is gauge invariant
- Get the correct N=8 SUGRA amplitude!
- This gauge is a special case of de Donder gauge with an unusual value for the gauge fixing parameter

\[ \alpha = -\frac{2\epsilon}{1 + \epsilon} \]

\[ \mathcal{L}^{(gf)} = \frac{\alpha}{2} \left( \partial_\nu h_\mu^\nu - \frac{1}{2} \partial_\mu h_\alpha^\alpha \right)^2 \]

Note: in usual de Donder gauge \( \alpha = -2 \)
- Graviton propagator in x-space, conf. gauge

\[ D_{\mu\nu,\mu'\nu'}(x) \sim \frac{\epsilon - 1}{\epsilon} \left[ \frac{1}{(-x^2)^{1-\epsilon}} \left( \eta_{\mu'}(\mu\eta_{\nu})_{\nu'} + \frac{\epsilon}{2(\epsilon - 1)^2} \eta_{\mu\nu} \eta_{\mu'\nu'} \right) + 2 \frac{1}{(-x^2)^{2-\epsilon}} x_{(\mu} \eta_{\nu)}(\nu' x_{\mu'}) \right] \]

- Gluon propagator in x-space, conf. gauge

\[ \Delta_{\mu\nu}^{\text{conf}}(x) \sim \frac{1 - \epsilon}{\epsilon} \frac{1}{(-x^2 + i\epsilon)^{1-\epsilon}} \left[ \eta_{\mu\nu} - 2 \frac{x_{\mu} x_{\nu}}{x^2} \right] \]

Inversion Tensor
Conclusions

• Mysterious relation between planar MHV amplitudes in N=4 SYM and light-like Wilson loops

• Why does this work? Dual conformal symmetry is insufficient to explain this, are there other symmetries?

• Unitarity for Wilson loop?

• Possible relations to world line formalism?

• What about other theories/non-MHV amplitudes?
  • 1-loop: Wilson loops insensitive to matter content of theory
  • 2 loops: Wilson loops in any SCFT identical
  • in N=1 SYM $\mathcal{M}_n^{(1)}$ depends on helicities
Conclusions cont’d

- **Iterative structure** in N=8 SUGRA amplitudes
  - IR divergences iterate completely
  - relatively simple finite remainder with uniform transcendentality
- **Wilson loop** reproduces almost the one-loop amplitude
  - IR divergent and finite parts come out correctly
  - cusps break the gauge invariance; can this be fixed?
- **Conformal gauge** gives the complete amplitude