A few aspects of quantum chaotic scattering

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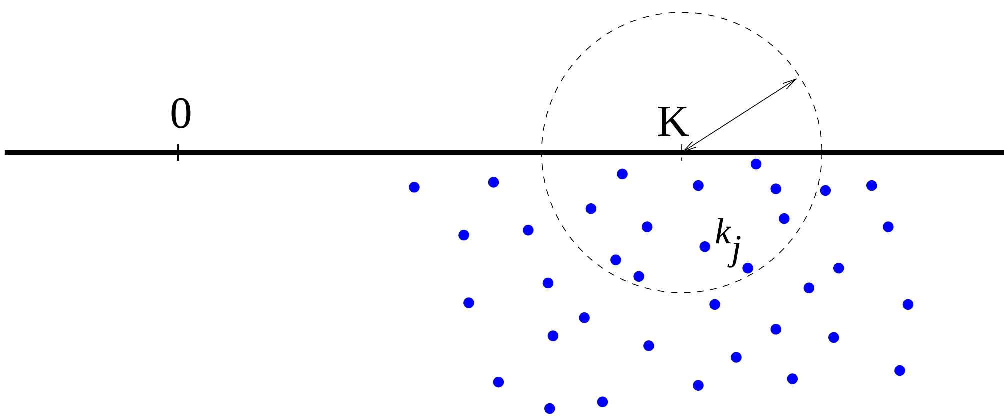
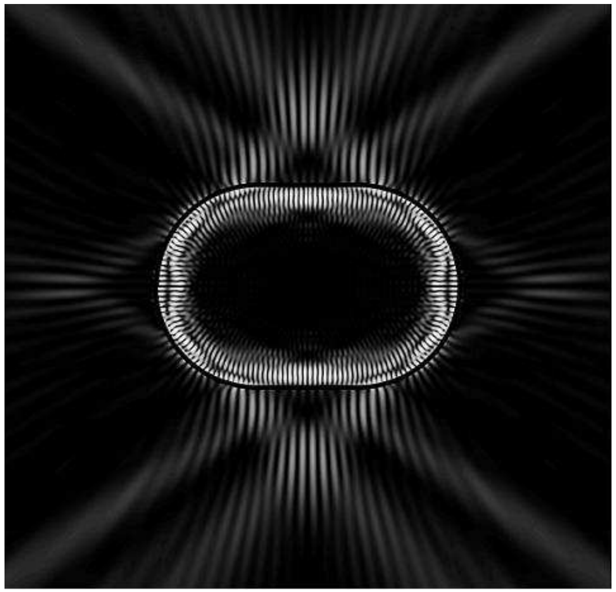




Right: resonant state for the dielectric stadium cavity, computed by C.Schmit.

Outline

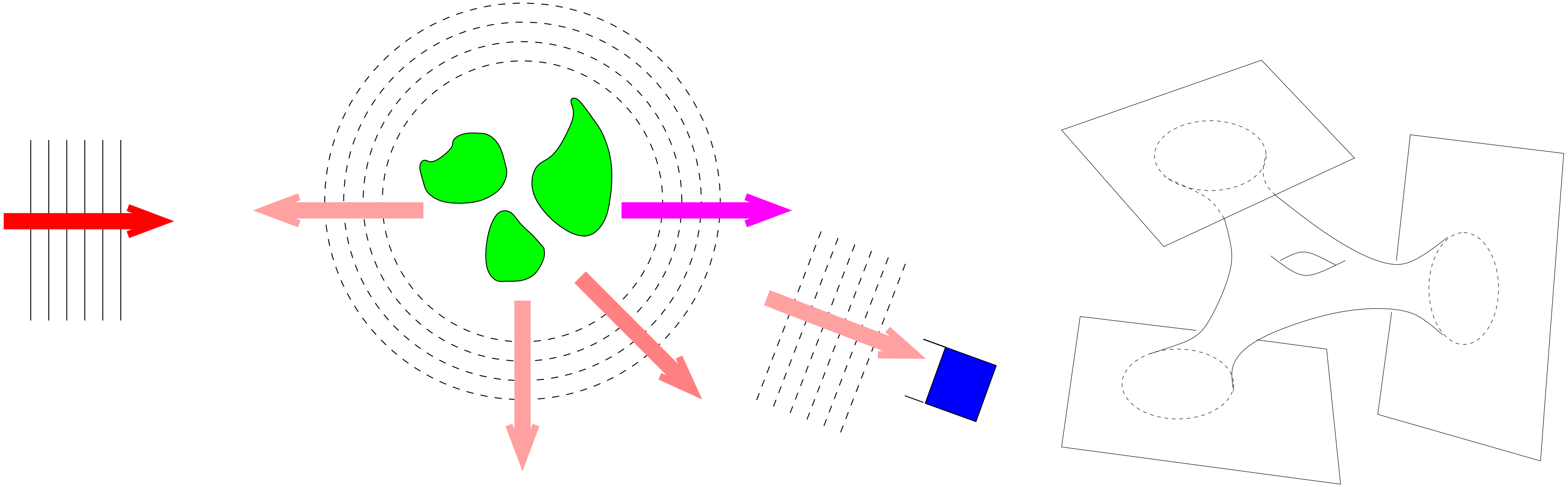
- scattering wave (quantum) systems $\rightsquigarrow$ (complex-valued) resonance spectra, metastable states

- Semiclassical (high-frequency) limit $\rightarrow$ need to understand the *ray dynamics*. Importance of the *set of trapped classical trajectories*.

- A toy model: open quantum maps
  - fractal Weyl law
  - resonance-free strip for filamentary trapped sets
  - phase space distribution of metastable states

- Another class of "leaky" quantum systems: *partially open* systems
  - clustering of decay rates near a *typical* value;
  - fractal Weyl laws

Euclidean scattering

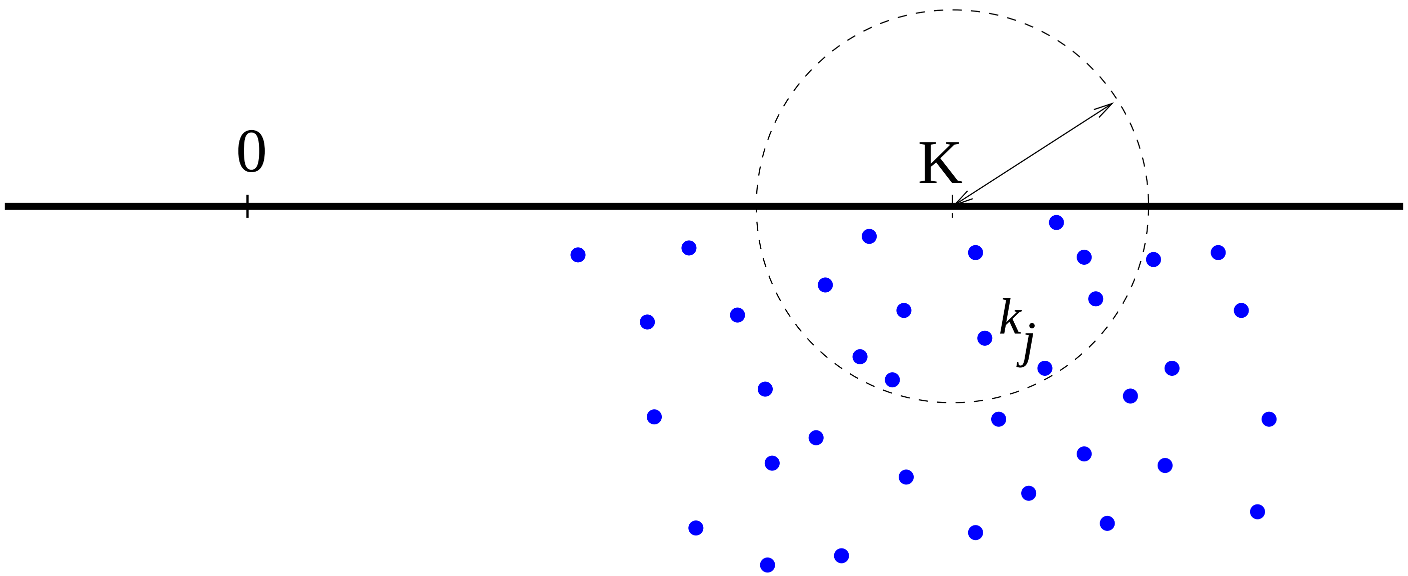


Scattering systems with hard obstacles/smooth localized potential/noneuclidean metric.

- classical dynamics: geodesic (or Hamiltonian) flow + reflection on obstacles. $Most$ $rays$ escape to infinity.

- quantum dynamics: wave or Schrödinger equation governed by $-\Delta_{out}$, resp. (or $P_\hbar = -\hbar^2\Delta + V(x)$)

Resonance spectrum



For any $E > 0$ the energy shell $\{(x, \xi),\ |\xi|^2 = E\}$ is **unbounded**, so $-\Delta_{out}$ has a **purely continuous** spectrum on $\mathbb{R}^+$.

- $(-\Delta_{out} - k^2)^{-1} : L^2_{comp} \to L^2_{loc}$ admits a meromorphic continuation from $\{\operatorname{Im} k > 0\}$ to $\{\operatorname{Im} k < 0\}$. Its poles $\{k_j\}$ (of finite multip.) are the **resonances** of $-\Delta_{out}$.

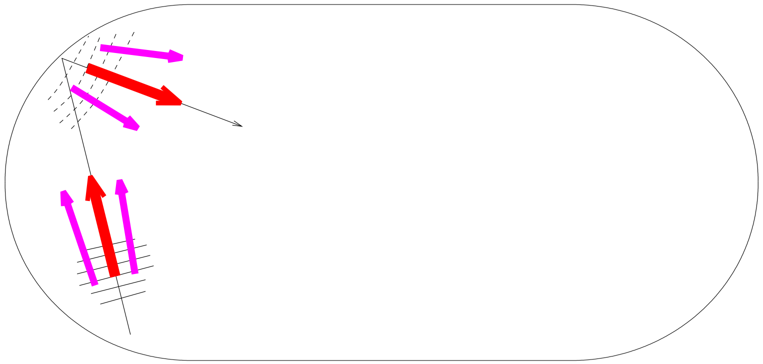
- Resonances = evals of a nonselfadjoint operator $-\Delta_{out,\theta}$ obtained from $-\Delta_{out}$ by a complex dilation (away from interaction zone)

- Each k_j is associated with a **metastable** (non-normalizable) state $\psi_j(x)$, with **decay rate** $\gamma_j = 2|\operatorname{Im} k_j| \longleftrightarrow$ **lifetime** $\tau_j = (2|\operatorname{Im} k_j|)^{-1}$.

  $\Longrightarrow$ long-living resonance if $\operatorname{Im} k_j = \mathcal{O}(1)$.

Semiclassical limit

We will focus on the high-frequency limit $\mathrm{Re}\, k \approx K \gg 1 \Rightarrow$ (micro)localized wavepackets propagate along *classical rays*.

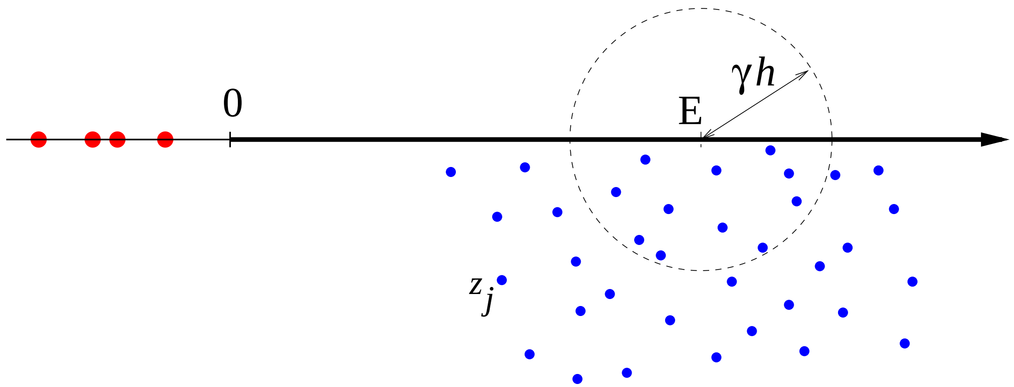


Take $\hbar_{eff} \stackrel{\text{def}}{=} K^{-1} \rightsquigarrow$ equivalent to study the resonances $\{z_i(\hbar)\}$ of $\hbar$-dependent operators

$$P_\hbar = -\hbar^2 \Delta_{out}, \quad \text{more generally} \quad P_\hbar = -\hbar^2 \Delta + V(x)$$

in a disk $D(E, \gamma\hbar)$ centered on a "classical energy" E.

High-frequency $\Longleftrightarrow$ semiclassical limit $\hbar \ll 1$.



Semiclassical limit (2)

Main questions we will consider in the semiclassical limit:

- distribution of long-living resonances ($|\operatorname{Im} z_j| = \mathcal{O}(\hbar)$)
- phase space localization of metastable modes $\psi_j(\hbar)$
- (time decay of the local intensity $|\psi(x,t)|^2$ (resolvent estimates))

Main idea: the distribution of long-living resonances depends on the properties of long classical trajectories.
Dispersion of the wave (due to the uncertainty principle) must also be taken into account.

$\longrightarrow$ relevance of the **set of trapped trajectories**:

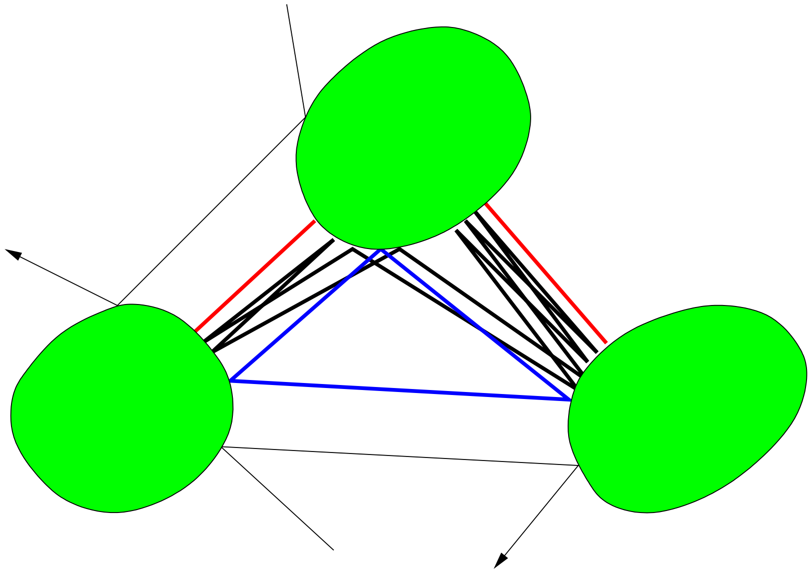
$$\Gamma^{\pm} = \{(q,p) \,:\, \phi^t(q,p) \not\to \infty, \quad t \to \mp\infty\}, \qquad \Gamma = \Gamma^+ \cap \Gamma^-$$

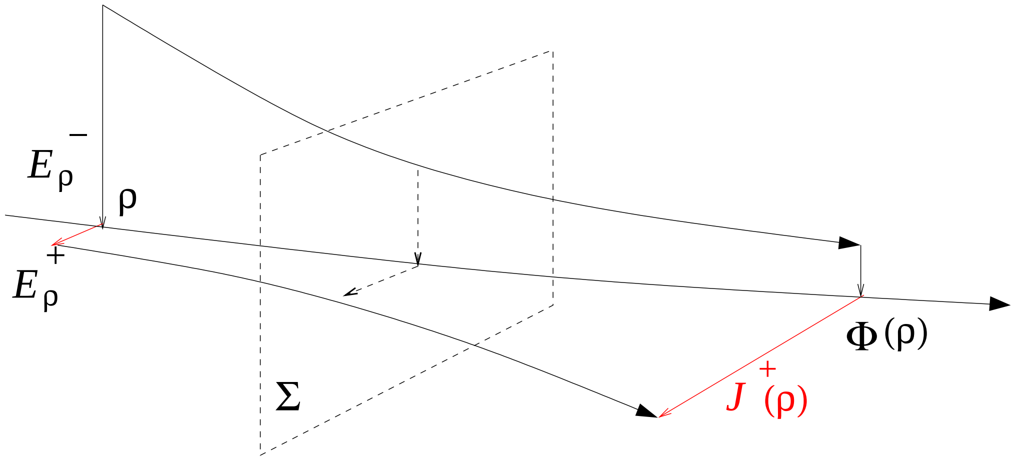
Long-living resonances represent *quantum mechanics living on* Γ.

Chaotic scattering

• We will focus on systems for which the classical flow on Γ is **strongly chaotic** (uniformly hyperbolic: Axiom A system). Such systems are not Liouville-integrable (no conserved quantity except E), but their long-time dynamics is well-understood.

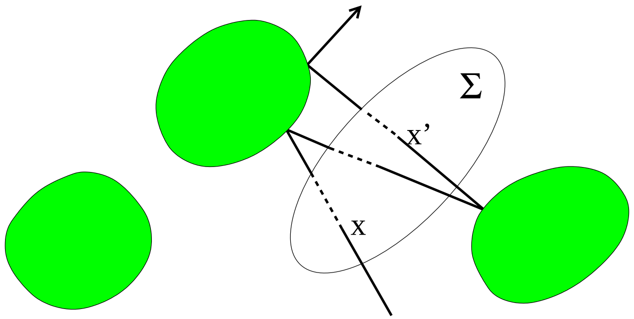
The trapped set Γ is a **hyperbolic repeller** with **fractal geometry**.

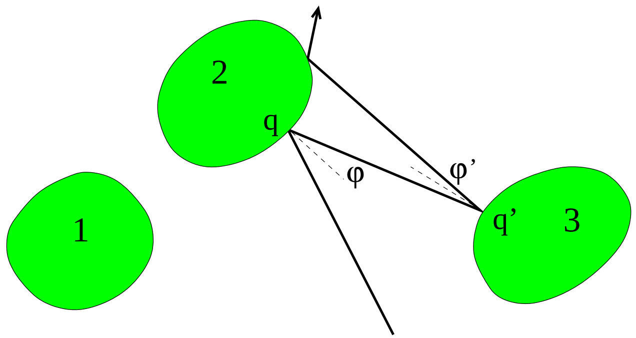




Semiclassical approach to *quantum chaos*: identify the appropriate classical-dynamical *tools* able to provide information on the quantum system.

A toy model: open maps





The ray dynamics can be analyzed through the **return map** κ through a **Poincaré section** Σ.

This map is defined on a subset $\Sigma' \subset \Sigma$, and preserves the induced symplectic form. It is an Axiom A homeomorphism on the trapped set $\Gamma \cap \Sigma$.

Ex: the bounce map on the obstables

$$(q, p = \sin\varphi) \mapsto \begin{cases} \kappa(q,p) & = (q', p') \\ \infty & \end{cases}$$

Generalization: consider an arbitrary symplectic chaotic diffeomorphism $\tilde{\kappa}$ on some compact phase space (e.g. the torus $\mathbb{T}^2$), and an arbitrary hole H through which particles escape "to infinity" $\rightsquigarrow$ open map $\kappa = \tilde{\kappa}_{\restriction \mathbb{T}^2 \setminus H}$.

A toy model: open quantum maps

How to "quantize" such a map κ? First, define quantum mechanics on $\mathbb{T}^2$:

- Hilbert space $\mathcal{H}_\hbar \equiv \mathbb{C}^N$, $N \sim \hbar^{-1}$
- quantization of observables: $f(q,p) \mapsto \mathrm{Op}_\hbar(f)$ (Pseudodifferential Operator)
- quantization of the diffeom $\tilde\kappa$ (various recipes): $U = U_\hbar(\tilde\kappa)$ unitary matrix (Fourier Integral Operator).

  Quantum-classical correspondence (until the Ehrenfest time $T_{Ehr} = \frac{|\log\hbar|}{\Lambda}$):

  $$U^{-t}\,\mathrm{Op}_\hbar(f)U^t = \mathrm{Op}_\hbar(f\circ\tilde\kappa) + \mathcal{O}(\hbar\, e^{\Lambda t}) \quad \text{[Egorov]}$$

  Equivalently, for a wavepacket $|q,p\rangle$, we have $U|q,p\rangle \approx |\tilde\kappa(q,p)\rangle$.

To open the "hole": apply a "projector" $\Pi = \mathrm{Op}_\hbar(\mathbb{1}_{\mathbb{T}^2\setminus H})$.

$\Longrightarrow$ open quantum map $\boxed{M_N(\kappa) = M_\hbar(\kappa) \stackrel{\text{def}}{=} \Pi \circ U_\hbar(\tilde\kappa)}$ $(N\times N$ subunitary)

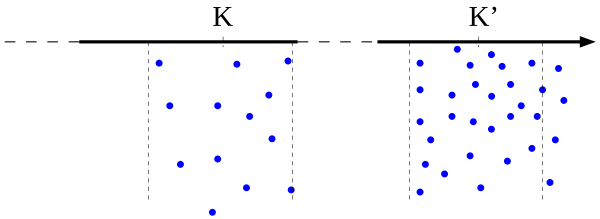
Correspondence with scattering resonances

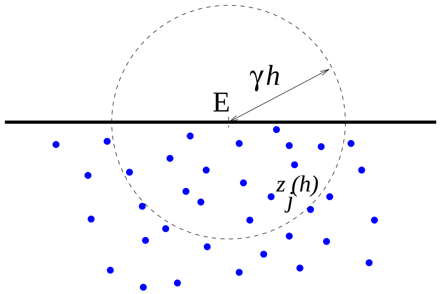
The spectrum $\{(\lambda_{i,N}, \psi_{i,N})\, i = 1, \ldots, N\}$ of the open map $M_\hbar(\kappa)$ should provide a good model for resonances of $P_\hbar$ (numerically much easier).

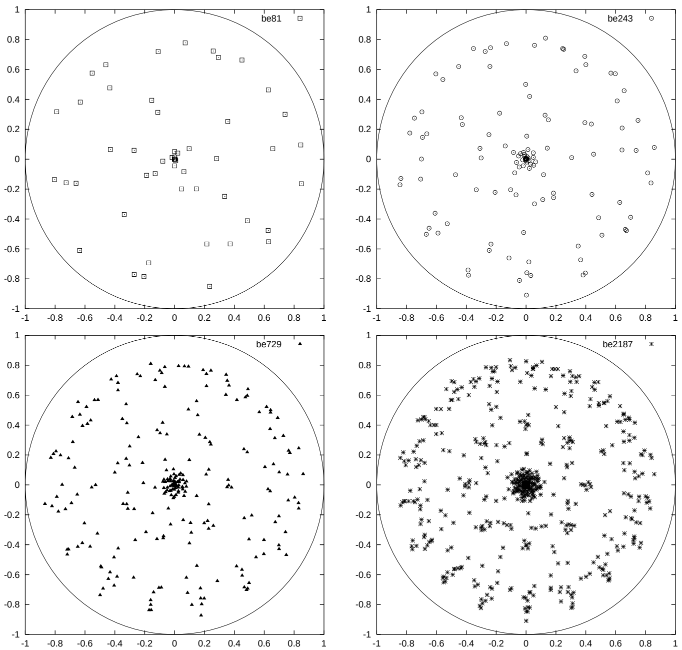
We expect the statistical correspondence:

$$\{\lambda_{i,N},\, i = 1, \ldots, N\} \longleftrightarrow \{e^{-iz_j(\hbar)/\hbar},\, |\operatorname{Re} z_j(\hbar) - E| \leq \gamma\hbar\}, \quad N \sim \hbar^{-1}$$

In particular, the decay rates $\{-2\operatorname{Im} z_j(\hbar)/\hbar\} \longleftrightarrow \{-2\log|\lambda_{i,N}|\}$.





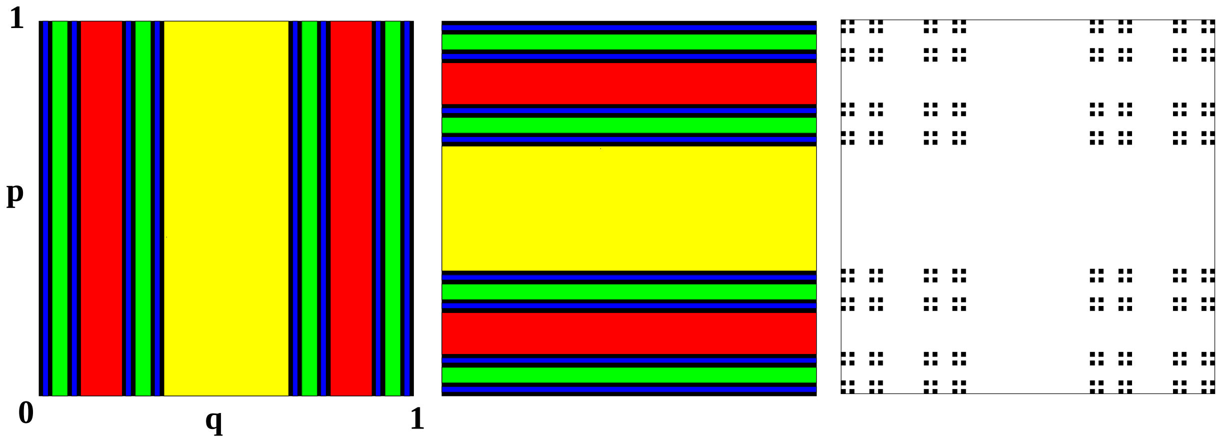


• To compute resonances of $P_\hbar$, one can actually construct a family of quantum maps $M_\hbar(z)$ associated with the Poincaré return map, such that $\{z_j(\hbar)\}$ are obtained as the roots of $\det(1 - M_\hbar(z)) = 0$ [N-Sjöstrand-Zworski'09?].

Example of an open chaotic map

Dig a rectangular hole in the 3-baker's map on $\mathbb{T}^2$

Advantage: the trapped sets $\Gamma^{(\pm)}$ are simple Cantor sets (simple symbolic dynamics)

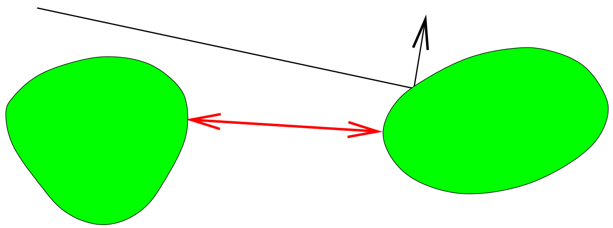


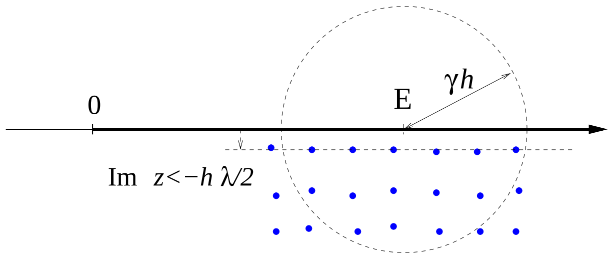
$$M_\hbar(B) = F_N^{-1} \begin{pmatrix} F_{N/3} & & \\ & 0 & \\ & & F_{N/3} \end{pmatrix}, \quad F_M = \text{discrete Fourier transform}$$

Fractal Weyl law

The *geometry* of the trapped set influences the semiclassical density of long-living resonances.

Ex: 2 convex obstacles $\Rightarrow$ $\Gamma=$ single unstable periodic orbit.
Quantum normal form $\rightsquigarrow$ quasi-lattice of resonances [IKAWA,GÉRARD,SJÖSTRAND,..]





How about a fractal repeller Γ?

Theorem. [Sjöstrand'90, Sjöstrand-Zworski'05] *In the semiclassical limit, the density of resonances is bounded from above by a fractal Weyl law*

$$\#\{j \;:\; |z_j(\hbar)-1| \leq \gamma\hbar\} = \mathcal{O}(\,\hbar^{-\nu}\,)\,, \quad \textit{resp.} \quad \#\{j \;:\; |\lambda_{j,N}| \geq c\} = \mathcal{O}(\,N^{\nu}\,)$$

where $\dim_{Mink}(\Gamma) = 2\nu + 1$ *(resp.* $= 2\nu$*).*

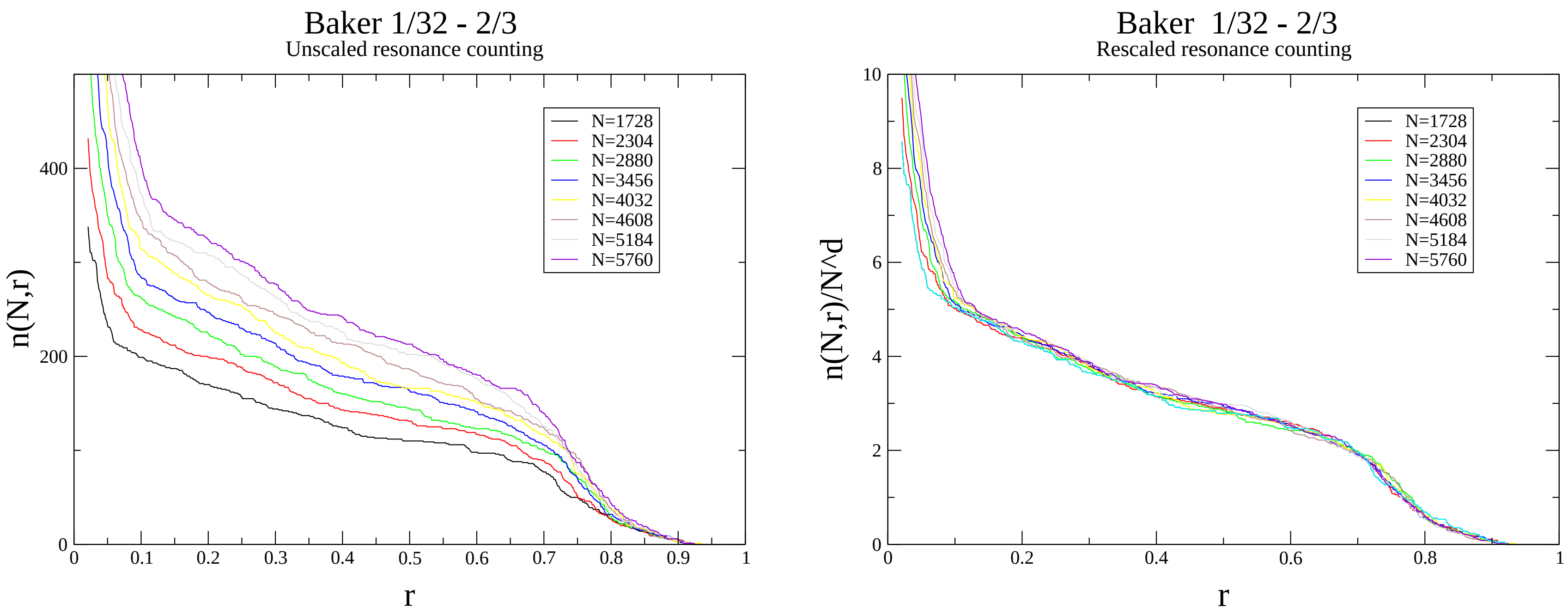
Main idea: after a suitable transformation, long-living resonant states "live" in a $\sqrt{\hbar}$-nbhd of Γ $\rightsquigarrow$ count the number of $\hbar^d$-boxes in this nbhd.

Conjecture: $= \mathcal{O}(\hbar^{-\nu})$ should be replaced by $\sim C_\gamma\, \hbar^{-\nu}$

Fractal Weyl law (2)

• Such a fractal Weyl law has been numerically confirmed for various systems.

Ex: an asymmetric open baker's map (ν known explicitly).



• This law was *proven* for an alternative *solvable* quantization of the open baker's map [N-Zworski'05].

• To understand the factor C_γ (shape of the curve), an ensemble of random subunitary matrices $(\Pi U)_{U \in COE}$ was proposed in [Schomerus-Tworzydlo'05]. Universal?

Resonance-free strip for "filamentary" repellers

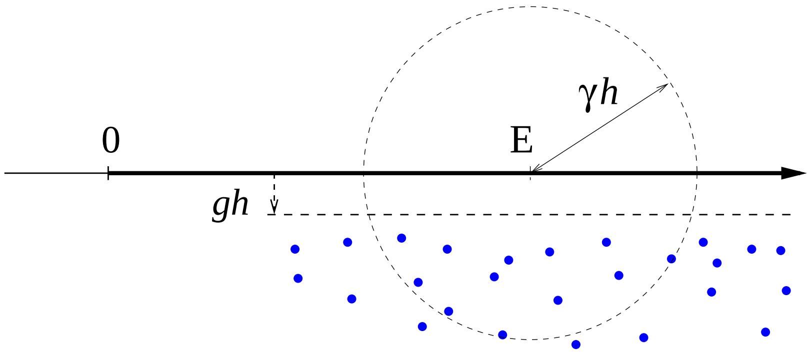
Another dynamical "tool" associated with the flow on Γ: the *topological pressure*

$$\mathcal{P}(s) = \mathcal{P}(-s \log J^+) \stackrel{\text{def}}{=} \lim_{t\to\infty} \frac{1}{t} \log \sum_{p:T_p \le t} J^+(p)^{-s}$$

"Compromise" between the *complexity* of the trapped set (# periodic orbits) and the *instability* of the flow along those orbits.

Properties: $\mathcal{P}(0) = h_{top}(\Phi^t_{\restriction\Gamma}) > 0$ and $\mathcal{P}(1) = -\gamma_{cl} < 0$ the classical decay rate.

Theorem. [Ikawa'88,Gaspard-Rice'89,N-Zworski'07] *Assume the topological pressure* $\mathcal{P}(1/2) < 0$, *and take any* $0 < g < -\mathcal{P}(1/2)$.
Then, for $\hbar > 0$ *small enough, the resonances* $z_j(\hbar)$ *close to* E *satisfy* $\operatorname{Im} z_j(\hbar) \le -g\,\hbar$.

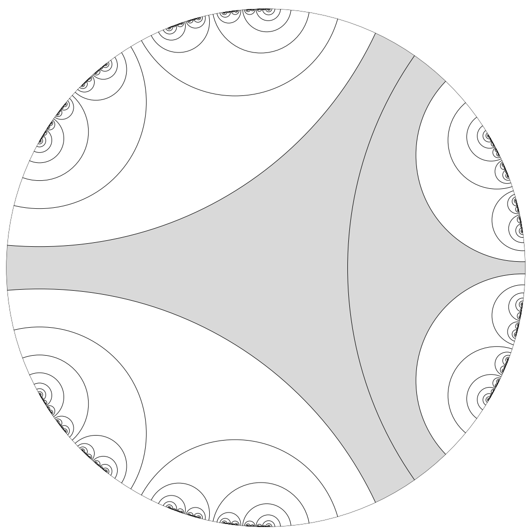


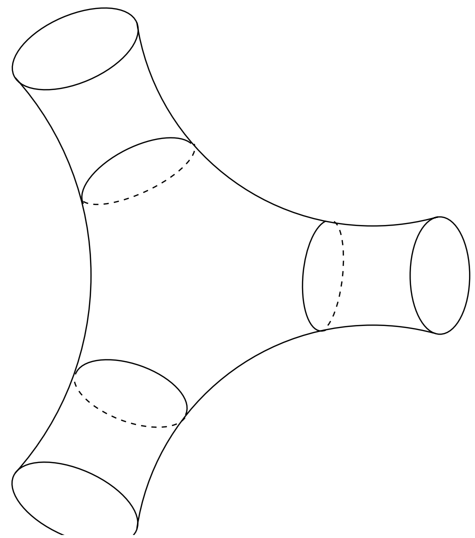
• In dimension $d = 2$, the dynamical condition $\mathcal{P}(1/2) < 0$ is equivalent with the geometrical condition $\dim(\Gamma) < 2$

A too thin repeller disperses the wave.

Analogous results on hyperbolic manifolds

$X = G\backslash\mathbb{H}^{n+1}$ convex co-compact (infinite volume). The trapped set Γ of the geodesic flow has dimension $2\delta + 1$, where δ is the dim. of the limit set $\Lambda(G)$, as well as the topological entropy of the flow.





Resonances $s(n-s) = \frac{n^2}{4} + k^2$ of Δ_X are given by the zeros of $Z_{Selberg}(s)$ (quantum resonances $\leftrightarrow$ Ruelle resonances)

[Patterson'76, Sullivan'79, Patterson-Perry'01]: all the zeros are in the half-plane $\mathrm{Im}\, k \leq \delta - n/2 = \mathcal{P}(1/2)$.

This upper bound can be slightly sharpened, and lower bounds for the gap can be obtained [Naud'06,'08]

Phase space distribution of metastable states

The metastable states $(\psi_j(\hbar))$ associated with long-living resonances have specific phase space distributions.

Consider a family of metastable (normalized) states $(\psi_{i_N})_{N\to\infty}$ of $M_N(\kappa)$ s.t. the corresponding resonances $|\lambda_{i_N}| \geq c > 0$. Up to extracting a subsequence, assume that (ψ_{i_N}) is associated with a semiclassical measure μ:

$$\forall f \in C^\infty(\mathbb{T}^2), \qquad \langle \psi_{i_N}, \mathrm{Op}_\hbar(f)\psi_{i_N}\rangle \xrightarrow{N\to\infty} \int_{\mathbb{T}^2} f\, d\mu\,.$$

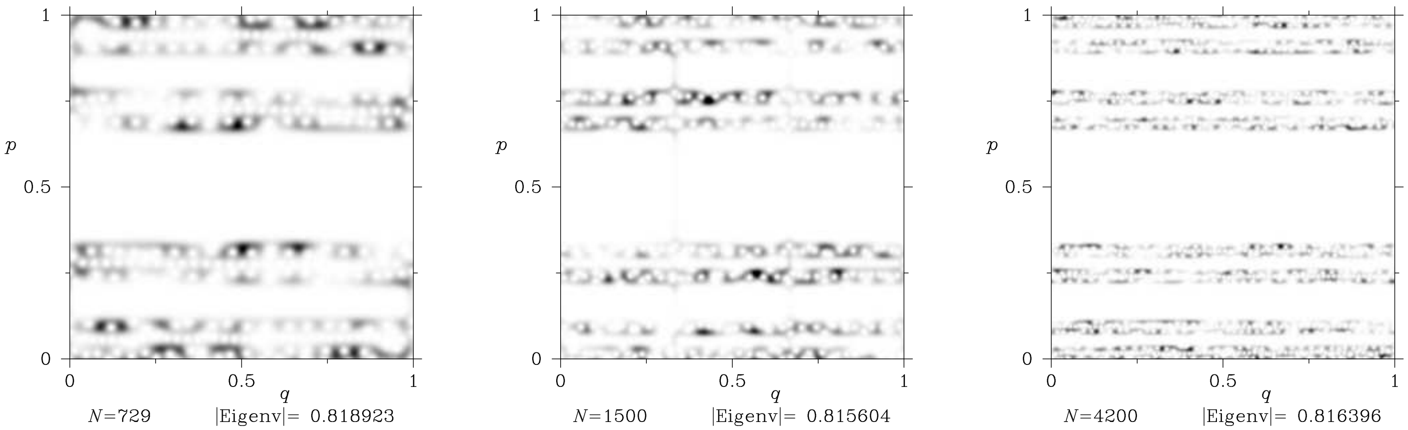
Then for some $\lambda \geq 0$ we have

$$|\lambda_{i_N}| \xrightarrow{N\to\infty} \lambda \quad \text{and} \quad \kappa^*\mu = \lambda^2\,\mu\,.$$

μ is a $\mathit{conditionally\ invariant\ measure}$ with decay rate λ^2.

Phase space distribution of metastable states (2)

Condit. invar. measures are easy to construct. They are supported on Γ^{+}.



Questions inspired by **quantum ergodicity** [N-Rubin'05, Keating-Novaes-Prado-Sieber'06]:

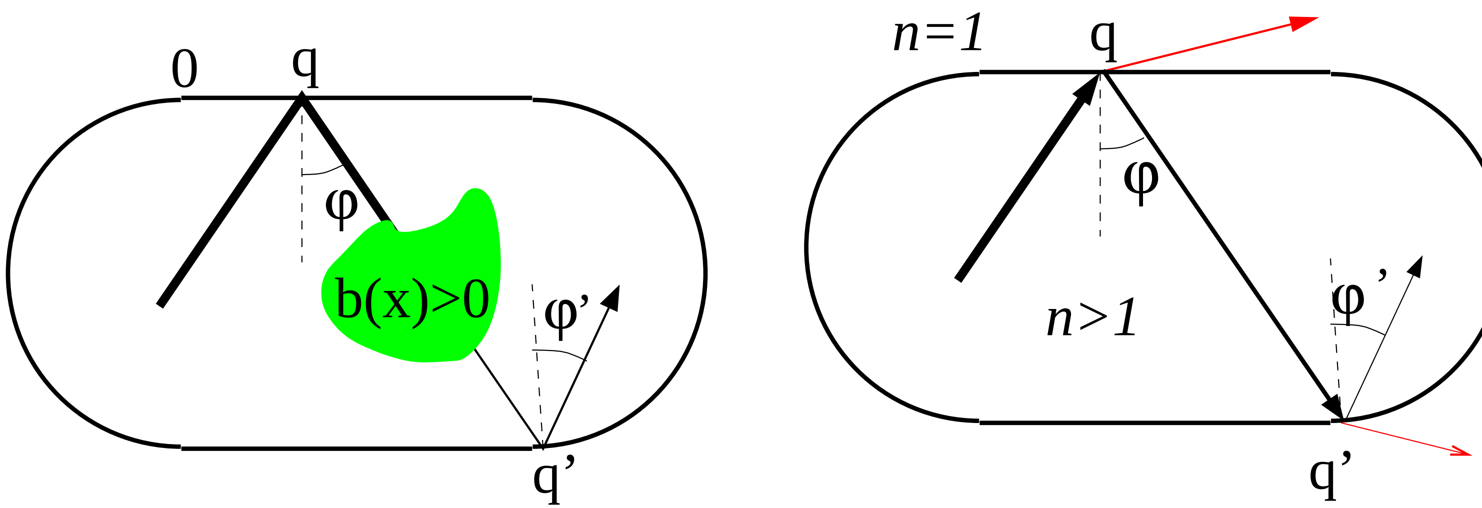
> For a given rate λ^2, which condit. invar. measures μ are favored (resp. forbidden) by quantum mechanics?

Very partial results for the *solvable* quantized open baker [Keating-Novaes-N-Sieber'08]:

- unique semiclassical measure at the edges of the nontrivial spectrum
- but not in the "bulk" of the spectrum (large degeneracies)

Partially open wave systems

Let us now consider systems for which rays do not escape, but *get damped*.



- Left: damped wave equation inside a closed cavity, $(\partial_t^2 - \Delta_{in} + b(x)\partial_t)\psi(x,t) = 0$, $b(x) \geq 0$ damping function
  $\rightsquigarrow$ spectrum of *complex eigenvalues* $(\Delta_{in} + k^2 + i\, b(x)\, k)\psi(x) = 0$

- Right: dielectric cavity. *Resonances* satisfy $(\Delta + n^2 k^2)\psi = 0$, with appropriate boundary conditions $\rightsquigarrow$ reflection+refraction of incoming rays (Fresnel's laws).

In both cases, the *intensity* ($\Leftrightarrow$ *energy*) of the rays is reduced along the flow.

$\longrightarrow$ Weighted ray dymamics.

Damped quantum maps

Starting from a diffeom. $\tilde\kappa$, one can cook up a damped quantum map:

$$M_\hbar(\tilde\kappa, d) \stackrel{\rm def}{=} \mathrm{Op}_\hbar(d) \circ U_\hbar(\tilde\kappa),$$

where $0 < \min|d| \leq |d(q,p)| \leq \max|d| \leq 1$ is a smooth damping function.

$\Rightarrow$ Bounds on the distribution of decay rates of $M_\hbar(\tilde\kappa, d)$:

- obvious: all N eigenvalues satisfy $\min|d| \leq |\lambda_{i,N}| \leq \max|d|$ (all resonances in a *strip*)

- Egorov $\Rightarrow M_\hbar^n \approx U^n \circ \mathrm{Op}_\hbar((d_n)^n)$, where we used the n-averaged weights

$$d_n(q,p) \stackrel{\rm def}{=} \big(\prod_{j=1}^n d(\tilde\kappa^j(q,p))\big)^{1/n}$$

$\Rightarrow$ all evals contained in the (often thinner) annulus $\min|d_\infty| \leq |\lambda_{i,N}| \leq \max|d_\infty|$.

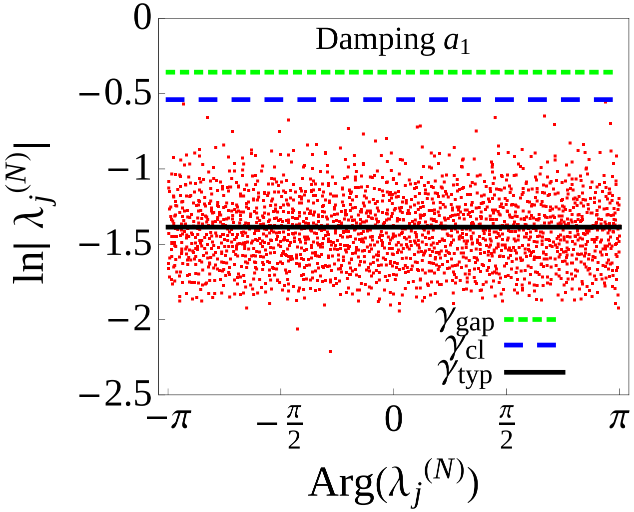
Taking the chaos into account: clustering of decay rates

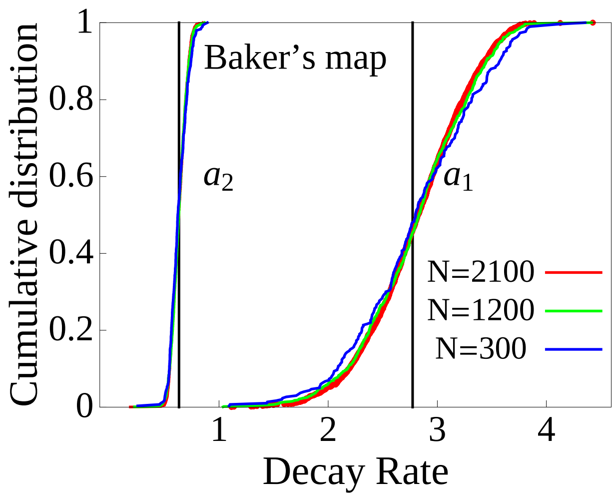
<u>Assume $\tilde{\kappa}$ Anosov</u> $\Rightarrow$ sharper bounds on the decay rate distribution.

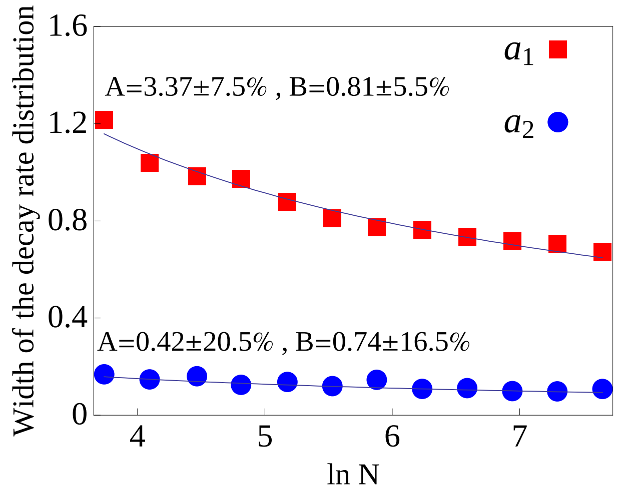
Ergodicity + Central Limit Theorem for d_n $\Rightarrow$ *almost all* the N evals satisfy

$$-2|\lambda_{i,N}| = \gamma_{typ} + \mathcal{O}((\log N)^{-1/2}),$$

where $\gamma_{typ} = -2 \int \log |d(q,p)| \, dq \, dp$ is the typical damping rate
$(|d_\infty(q,p)| = e^{-\gamma_{typ}/2}$ almost everywhere) [SJÖSTRAND'00,N-SCHENCK'08].







Is the width of the distribution really $\mathcal{O}((\log N)^{-1/2})$? (OK for the solvable quantized baker's map).

Fractal Weyl law in the distribution tails

Large deviation estimates for $d_n \Rightarrow$ *fractal upper bounds* for the density of resonances away from γ_{typ}.

Theorem. [Anantharaman'08,Schenck'08]

$$\forall \alpha \geq 0, \quad \#\{i\,:\,-2\log|\lambda_{i,N}| \approx \gamma_{typ} + \alpha\} \leq C_\alpha\, N^{f(\alpha)}$$

$f(\alpha) \in [0,1] \leftrightarrow$ *the rate function for* d_n.

Solvable baker's map: the above bound is generally not sharp.

One can also bound the decay rates using an adapted topological pressure.

Theorem. [Schenck'09] *For any $\epsilon > 0$ and any large enough $N \sim \hbar^{-1}$,*

$$-2\log|\lambda_{i,N}| \geq -2\mathcal{P}\big(-\frac{1}{2}\log J^+ + \log|d|\big) - \epsilon$$

In some situations, the RHS is larger than $-2\log\max|d_\infty|$.

Phase space distribution of metastable states (3)

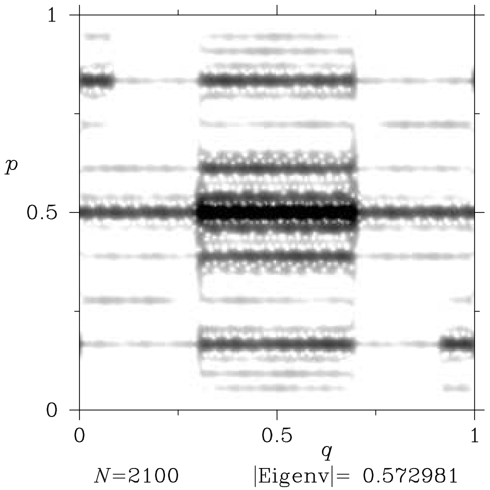
Partially open system [Asch-Lebeau'00, N-Schenck'09]: semiclassical measures associated with metastable states satisfy

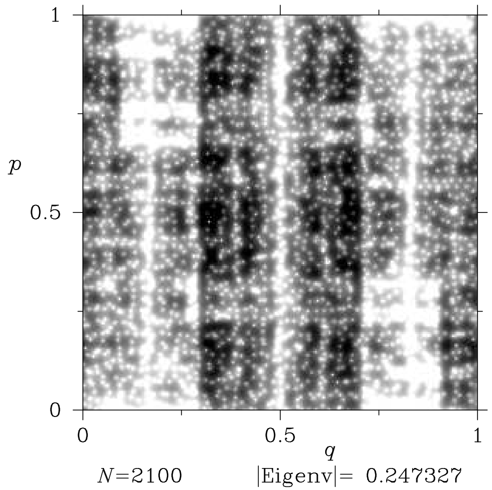
$$|d|^2 \times \tilde{\kappa}^* \mu = \lambda^2 \mu.$$

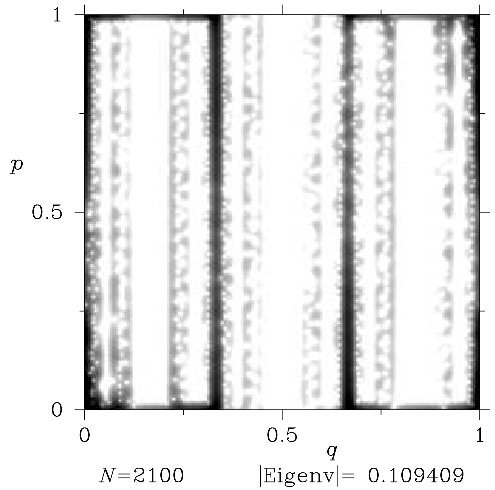
Such condit. invar. measures are more difficult to classify than in the fully open case.

Several numerical studies for a chaotic dielectric cavity [Wiersig,Harayama,Kim..]

Examples of Husimi measures for a partially open 3-baker.







Work in progress...