Fractal Weyl law and conductance fluctuations for the quantized open baker’s map

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Quantum (chaotic) scattering

In a **scattering** situation (**open** system), the classical motion is **unbounded**.

Famous examples:
- $n$-disks scatterer ($n \geq 3$) [Gaspard+, Cvitanovic+...], smooth potentials of compact support, Coulomb scatterers [Knauf-Klein]
- free motion on a **noncompact Riemann surface** of negative curvature [Selberg...]
- conduction through an **open chaotic cavity** [Weidenmüller, Blümel-Smilansky,Beenakker+,...].
- open quantum graphs [Kottos-Smilansky...], open chaotic maps.

One can study the scattering through the (unitary) **scattering matrix** $S(E)$, which connects incoming waves to outgoing waves.

If the openings consist in two **leads**, $S(E)$ splits into 4 blocks (transmission vs. reflection):

$$S(E) = \begin{pmatrix} r_{11} & t_{12} \\ t_{21} & r_{22} \end{pmatrix}.$$
Resonances of open quantum systems

In a scattering situation, the Hamiltonian $\hat{H}_h = p^2 + V(q)$ has a purely continuous spectrum: no bound states.

But the resolvent $(z - \hat{H}_h)^{-1}$ may often be continued meromorphically across the positive axis; in the second sheet it has discrete poles \{\(z_j\)\}: quantum resonances of $\hat{H}_h$. These are also poles of $S(E)$.

Each $z_j = E_j - i\Gamma_j$ is associated with a metastable (non-normalizable) state with lifetime $\tau_j = \frac{\hbar}{\Gamma_j} \implies$ finite lifetime if $\Gamma_j = O(\hbar)$.

2 Questions in the semiclassical regime:

- Distribution of resonances (with $\Gamma_j = O(\hbar)$).

- In the 2-lead situation, description of the conduction through the cavity: distribution of the eigenvalues of $t_{21}^* t_{21}$ (transmission eigenvalues).
  
  * Landauer formula: conductance $g \propto \text{tr}(t_{21}^* t_{21})$.
  
  * Shot noise due to quantum interferences: $P \propto \text{tr}\left\{t_{21}^* t_{21} - (t_{21}^* t_{21})^2\right\}$.  

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Reminder: Weyl law for a closed quantum system

If the motion generated by $H = p^2 + V(q)$ on $\mathbb{R}^d \times \mathbb{R}^d$ is bounded near energy $E$, the operator $\hat{H}_h$ admits near $E$ a discrete spectrum.

In the semiclassical limit, the number of eigenenergies near some non-degenerate energy $E$ is given by the Weyl law [Weyl...Ivrii]:

$$\# \left\{ \text{Spec}(\hat{H}_h) \cap [E - Ch, E + Ch] \right\} \sim C \text{Liouv}(\{H(x) = E\}) h^{1-d}$$

Similarly, a canonical map on the torus phase space, $M : \mathbb{T}^2 \to \mathbb{T}^2$, is quantized into a $N$-dim. unitary operator $\hat{M}_N$ on $\mathcal{H}_N$ ($N = h^{-1}$), with $N$ eigenphases $e^{i\theta_j} \equiv e^{-iE_j/h}$ on the unit circle [Berry+...]

If $\text{Leb}(\text{periodic points}) = 0$, eigenphases are distributed uniformly on the unit circle [Zelditch, Bouzouina-De Bièvre, Marklof-O’Keefe]:

$$\# \left\{ \text{Spec}(\hat{M}_N) \cap [\varphi_1 \leq \theta \leq \varphi_2] \right\} \sim \frac{\varphi_2 - \varphi_1}{2\pi} N.$$
Take a scattering potential $V(q)$ with chaotic dynamics at energy $E$, such that the sets $K_+(E)$ (resp. $K_-(E)$) of points of energy $E$ which never escape at $n \to +\infty$ (resp. $n \to -\infty$) have zero Lebesgue measure, and are fractal.

The trapped set at energy $E$, $K_E = K_{E,+} \cap K_{E,-}$, is a fractal set, with dimension $\dim_{Mink}(K_E) = 2\mu_E + 1$ ($\mu_E < d-1$).

**Conjecture:** the semiclassical density of resonances of $\hat{H}_\hbar$ is given by the fractal Weyl law

$$\forall r > 0, \quad \# \{ z_j : |z_j - E| < r \hbar \} \sim C(r) \hbar^{-\mu_E}$$

Except in trivial cases ($K(E)$ of dimension $1$), only the upper bound has been proven so far [Sjöss., Zwo., Guil....].

Numerics seem to confirm this conjecture.
Classical and quantum open maps

As a toy model for scattering, we consider a **canonical map with a hole** on the torus, $M : \mathbb{T}^2 \setminus \text{hole} \to M(\mathbb{T}^2 \setminus \text{hole})$.

One can extend $M$ to an invertible map $\tilde{M} : \mathbb{T}^2 \to \mathbb{T}^2$, but at each time step, *send all points in the hole to infinity*. (Interpretations: “ionization” of the particle/*open quantum dot*.)

**Quantization** of such an open map:

- the closed map $\tilde{M}$ is quantized into a unitary matrix $\hat{U}_N$ on $\mathcal{H}_N$, $N = \hbar^{-1}$.
- the escape through the hole is quantized by a *phase space projector* $\hat{\pi}_{N,\text{hole}} = I_N - \hat{\pi}_{N,\text{Int}}$.
- the quantum open map is then defined as $\hat{M}_N = \hat{U}_N \hat{\pi}_{N,\text{Int}}$, a subunitary matrix on $\mathcal{H}_N$.

$\text{rank}(\hat{M}_N) \approx \text{rank}(\hat{\pi}_{N,\text{Int}}) \approx (1 - |\text{hole}|)N$, and the matrix $\hat{M}_N$ is non-normal.
Advantage of the compact phase space: no more continuous spectrum, no need of analytic continuation of the resolvent.

Resonances $\leftrightarrow$ eigenvalues $\lambda_j \equiv e^{-z_j/\hbar}$ of the quantized open map $\hat{M}_N$. 

- Analogous conjecture for an open chaotic map $M$, with future trapped set $K_+$ of dimension $\mu$: we count their number in annuli $A_r = \{ \lambda : r < |\lambda| \leq 1 \}$.

**Conjecture:** $\forall r > 0$, $\# \left\{ \text{Spec}(\hat{M}_N) \cap A_r \right\} \sim C(r) N^\mu$.

[Casati+, Beenakker+, Schomerus+] studied the open kicked rotator. The profile function $C(r)$ may be universal (RMT).

We will consider the open baker's map $B$. 
The map is defined as follows on a fraction of $\mathbb{T}^2$:

$$
B(q,p) = \begin{cases} 
(3q, p/3) & \text{if } q < 1/3 \\
\infty & \text{if } 1/3 < q < 2/3 \\
(3q - 2, (p + 2)/3) & \text{if } q > 1/3 
\end{cases}
$$

Uniform Lyapunov exponent $\lambda = \log 3$. Simple symbolic dynamics (ternary decomposition of $q, p$) $\implies$ exponential mixing on the (future) trapped set $K_+$, a Cantor set of dimension $\mu = \frac{\log 2}{\log 3}$. 
Construction of the future trapped set $K_+$

The colored strips represent points going to $\infty$ at times $n = 1, 2, 3, \ldots$. $K_+$ is the remaining set, a fractal of dimension $\frac{\log 2}{\log 3}$. 
Quantum open baker’s map

For $N$ a multiple of 3, this map can be quantized into a subunitary matrix $\hat{B}_N$ on $\mathcal{H}_N$ [Balazs-Voros, Saraceno-Vallejos]. In the position basis $\{|q_j\rangle \in \mathcal{H}_N, j = 0, \ldots, N - 1\}$

$$\hat{B}_N = F_N^{-1} \begin{pmatrix} F_{N/3} & 0 \\ 0 & F_{N/3} \end{pmatrix}, \quad F_N = \text{Discrete Fourier Transf.}$$

The entries are maximal on the “tilted diagonals”.

![Baker (BV) and Baker (W) diagrams](image-url)
Resonances of the quantized open baker’s map

Conjecture: $\forall r > 0, \quad \# \left\{ Spec(\hat{B}_N) \cap A_r \right\} \sim C'(r) N^\frac{\log 2}{\log 3}.$

Numerically, it seems to work, especially along geometric sequences $N_k = 3^k N_o$. When $N \to 3N$, the number of resonances is $\pm$ multiplied by 2.

Similar evidence for the open kicked rotator [Schomerus-Tworzydło '04].
Left: raw counting of resonances $> r$. Right: rescaling by $N^{-\frac{\log 2}{\log 3}}$. 
Even eigenvalues of open 3-baker

Counting eigenvalues in logarithmic scale

\[ \log(N) \]

\[ \log(n(N,r)) \]

N=200*3^k, r=0.5
N random, r=0.5
N=27x3^k, r=0.5
N=27x3^k, r=0.15
N=27x3^k, r=0.03
theoretical slope
Idea of proof for the upper bound

Heuristics [Schom.-Twor.]: semiclassical estimate of the dimension of the generalized quasi-kernel (cf large 0-Jordan blocks).

To each connected strip $S_{t,j}$ escaping at time $t$, one associates a subspace $\mathcal{H}_{t,j}$ of dimension $\approx N|S_{t,j}|$ which is (almost) killed by $\hat{B}_N^t$.

At the Ehrenfest time $\tau_E \approx \frac{\log N}{\lambda}$, the strips have area $|S_{\tau_E,j}| \approx \frac{1}{N}$, the minimal possible.

Mathematically, control of $\hat{B}_N^t$ until $\tau_E$ is OK away from discontinuities of $B^t$.

Using coherent states we may construct $\mathcal{H}_{t,j}$ of dim. $N|S_{t,j}| - N^\epsilon$.

$\implies \hat{B}_N^{\tau_E}$ has $\sum_{t \leq \tau_E} \sum_j \dim(\mathcal{H}_{t,j})$ very small singular values

$\implies$ upper bound $\forall r > 0$, $\# \left\{ \text{Spec}(\hat{B}_N) \cap A_r \right\} \leq O(N^{\mu+\epsilon})$. 
To obtain analytical results, we (brutally) replace $\hat{B}_N$ by a “skeleton matrix” $\hat{T}_N$.

$$\hat{T}_9 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \omega & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \omega^2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \omega \end{pmatrix}, \quad \omega = e^{2\pi i/3}.$$  

$\hat{T}_N$ had been proposed before [Schack-Caves, Saraceno] as a toy quantization of the baker. Appears also as a quantum binary graph [Tanner].

**BUT:** $\hat{T}_N$ does **NOT** transport Gaussian coherent states according to $B$, but rather through the multivalued $(q,p) \mapsto B(q,p) + (0, j/3)$, $j = \{0, 1, 2\}$ (aliasing). The future trapped set is still $K_+$. 

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Walsh Fourier transform and the toy model \((N = 3^k)\)

In the case \(N = 3^k\), the triadic decomposition \([0, 3^k - 1] \ni j \equiv \epsilon_1 \epsilon_2 \cdots \epsilon_k\), \(\epsilon_\ell \in \{0, 1, 2\}\) leads to the tensor-product decomposition \(\mathcal{H}_N \equiv (\mathbb{C}^3)^\otimes n\), where the position eigenstates are tensor products:

\[
|q_j\rangle = e_{\epsilon_1} \otimes e_{\epsilon_2} \otimes \cdots \otimes e_{\epsilon_k} \quad (\{e_0, e_1, e_2\} \text{ the canonical basis of } \mathbb{C}^3).
\]

The Walsh-Fourier transform \(W_N\) is defined as follows:

\[
(W_N)_{jj'} = 3^{-k/2} \exp\left(-2i\pi \frac{\epsilon_1 (jj')}{3}\right),
\]

whereas \((F_N)_{jj'} = 3^{-k/2} \prod_{\ell=1}^{k} \exp\left(-2i\pi \frac{\epsilon_\ell (jj')}{3^\ell}\right)\).

One can then check that the toy model is a Walsh-quantized baker’s map:

\[
\hat{T}_N = W_N^{-1} \begin{pmatrix}
W_{N/3} & 0 \\
0 & W_{N/3}
\end{pmatrix}.
\]
Spectral analysis of the toy model \((N = 3^k)\)

\(\hat{T}_N\) acts nicely on tensor-product states:

\[
\hat{T}_N(v_1 \otimes \cdots \otimes v_k) = v_2 \otimes \cdots v_k \otimes G_3 v_1, \quad G_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & \bar{\omega} \\ 1 & 0 & \omega \end{pmatrix}.
\]

Using this, we can analytically compute the whole spectrum of \(\hat{T}_N\) in terms of that of \(G_3\):

\[Spec(G_3) = \{(0, e_2), (\lambda_+, v_+), (\lambda_-, v_-)\} \text{ with } 1 > |\lambda_+| > |\lambda_-| > 0.\]

The generalized kernel of \(\hat{T}_N\) has dimension \(3^k - 2^k\), and the \(2^k\) nonzero eigenvalues are forming a lattice

\[
\lambda_{m,p} = e^{2i\pi m/k} \lambda_+^{1-p/k} \lambda_-^p, \quad p = 1, \ldots, k - 1, \quad j = 0, \ldots, p - 1,
\]

with large degeneracies around the radius \(r_0 = \sqrt{|\lambda_+\lambda_-|}\).

In the semiclassical limit, this spectrum satisfies the fractal Weyl law

\[
\# \left\{ Spec(\hat{T}_{N=3^k}) \cap A_r \right\} = \theta(r_0 - r) 2^k + o(2^k).
\]

\(C(r) = \theta(r_0 - r)\) is very non-universal: \(\hat{T}_N\) is far from a random matrix.
Spectrum of the toy model

Nontrivial spectrum of $\hat{T}_N$ for $N = 3^{10}$ (o) and $N = 3^{15}$ (x). Right: logarithmic repr.
Conduction through a Walsh-quantized baker

As a model for transmission through a mesoscopic cavity, we consider a closed 4-baker’s map, and open it by 2 leads.

\[ \sim \sim \text{Walsh quantization } \hat{U}_N \text{ for } N = 4^k \text{ The two leads } \sim \sim \hat{\pi}_{N,1}, \hat{\pi}_{N,2}. \]

In this situation, the \( 4^{k-1} \times 4^{k-1} \) transmission matrix \( \hat{t}_{21}(\vartheta) \) is given by

\[
\hat{t}_{21}(\vartheta) = \hat{\pi}_{N,2} \sum_{n \geq 0} e^{-i n \vartheta} \hat{U}_N(\hat{\pi}_{N,\mathrm{Int}} \hat{U}_N)^n \hat{\pi}_{N,1} \quad (\vartheta = \text{quasi-energy}).
\]

In the semiclassical limit, we can compute both the conductance and the shot noise power (2 first moments of the eigenvalues \( \{T_i\} \) of \( t_{21}^* t_{21} \)).

- \( 4^{k-1} - 2^{k-1} \) classical conduction channels, half are fully reflected, half are fully transmitted. Average conductance \( \frac{1}{2} \), noiseless.

- \( 2^{k-1} \) quantum channels, with eigenvalue \( 0 < T_i < 1 \). Average conductance \( \frac{1}{2} \). Average noise power \( \frac{11}{80} \).

Remark: prediction of RMT for the average noise power: \( \frac{1}{8} \) [Weidenmuller...].
Perspectives

The toy model $\hat{T}_N$ is a sort of semiquantum map, intermediate between
– the (fully) quantum map $\hat{B}_N$.
– the classical transfer matrix ($\omega \to 1$) on a Markov partition, with spectrum the Ruelle-Pollicott resonances.
It does not see the discontinuities of $B$ (no diffraction), but leads to aliasing.

→ Can one get analytic results (upper/lower bounds) for $\hat{B}_N$?
Is the for the “profile function” $C(r)$ for $\hat{B}_N$ universal?

• Right (resp. left) eigenfunctions of $\hat{B}_N$ are localized on $K_-$ (resp. $K_+$).
Do their Husimi functions converge to any invariant measure on $K_\mp$ (cf [Chernov])?

• Conduction problem: full distribution of the $\{T_i\}$? Is $S(\vartheta)$ more random than $\hat{T}_N$?
3-bump chaotic potential in 2 dimensions
Free motion on a co-compact quotient of $\mathbb{H}$.

Left: the grey area is the union of 2 fundamental domains representing the quotient $\mathbb{H}/\Gamma$ ($\Gamma$ a Schottky group). This quotient is a manifold of infinite area, with 3 “hyperbolic leads” (right).

Except for a **fractal subset**, all geodesics start and/or end in a lead: chaotic scattering.

The resonances of $-\hbar^2 \Delta_{LB}$ are simply related with the zeroes of the Selberg zeta function.
Closed vs. open quantum cavity (dot)

Left: closed cavity. Right: open cavity, connected with 2 leads.