Steep tilings and sequences of interlaced partitions

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Joint work with Guillaume Chapuy and Sylvie Corteel
arXiv:1407.0665

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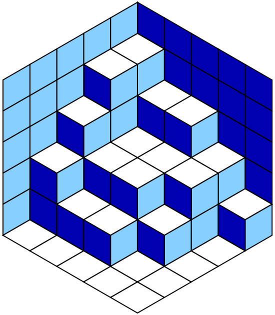
Integrability and Representation Theory seminar
University of Illinois at Urbana-Champaign
24 September 2014

Introduction

Rhombus tiling



Plane partition

4	3	3	2	1
4	2	2	2	
3	2	2	1	
2	1	1		

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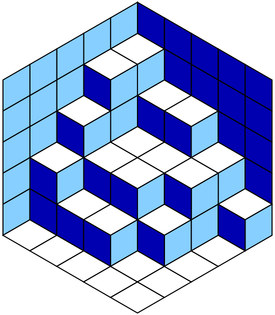

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Sequence of interlaced partitions

$$2 \prec \begin{matrix} 3 \\ 1 \end{matrix} \prec \begin{matrix} 4 \\ 2 \\ 1 \end{matrix} \prec \begin{matrix} 4 \\ 2 \\ 2 \end{matrix} \succ \begin{matrix} 3 \\ 2 \\ 1 \end{matrix} \succ \begin{matrix} 3 \\ 2 \end{matrix} \succ 2 \succ 1$$

Introduction

An (integer) partition λ is a finite non-increasing sequence of integers

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_\ell > 0$$

(By convention we set $\lambda_i = 0$ for $i \geq \ell$.)

We say that λ and μ are (horizontally) interlaced, and denote $\lambda \succ \mu$, iff

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Plane partitions correspond to sequences of interlaced partitions:

$$\ldots \lambda^{(-2)} \prec \lambda^{(-1)} \prec \lambda^{(0)} \succ \lambda^{(1)} \succ \lambda^{(2)} \succ \ldots$$

with $\lambda^{(i)} = \emptyset$ for $|i|$ large enough.

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How about tilings made of dominos instead of rhombi?

Motivations

In general:

- statistical mechanics: rhombus/domino tilings = dimer model on honeycomb/square lattice
- enumerative combinatorics: beautiful enumeration formulas
- probability theory: determinantal correlations, limit shape phenomena, interesting limiting processes related to random matrices
- algebraic geometry: Donaldson-Thomas theory

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For our specific work: understand precisely the connection between domino tilings and interlaced partitions, implicitly hinted at in works of Johansson, Borodin, etc. Have fun with "vertex operators" (a recreation after a reading group on the works from the Kyoto school: solitons, infinite dimensional Lie algebras and all that).

Outline

1. Steep tilings
2. Bijection with sequences of interlaced partitions
3. Enumeration via the vertex operator formalism

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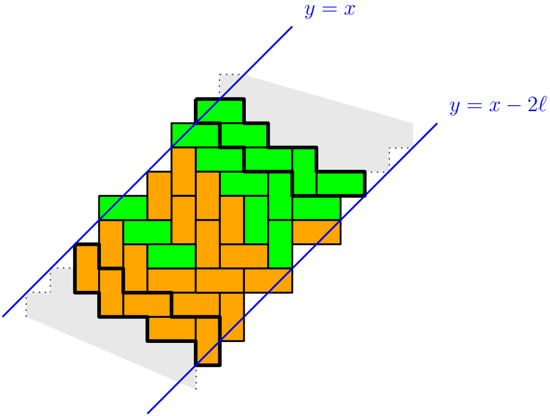
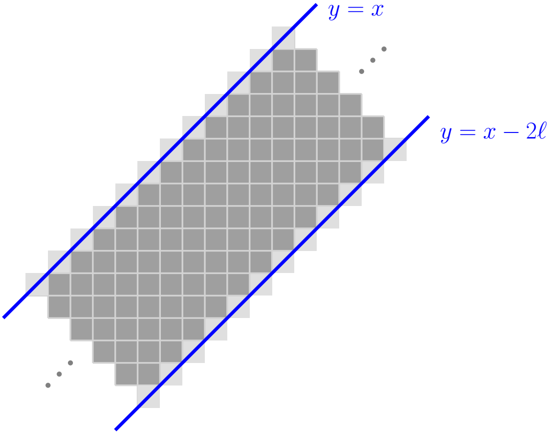
Steep tilings



- A domino tiling of the oblique strip $x - 2\ell \leq y \leq x$
- Steepness condition: we eventually find only north or east dominos in the NE direction, south or west in the SW direction.

Steep tilings



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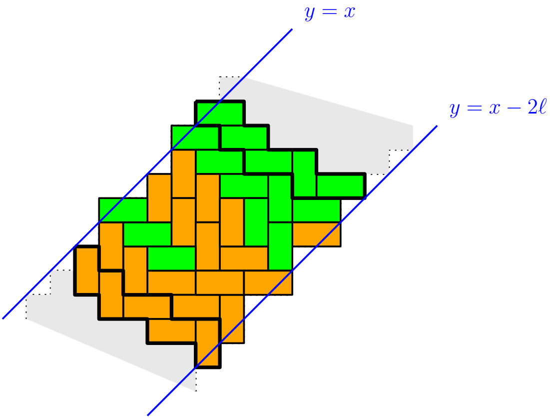

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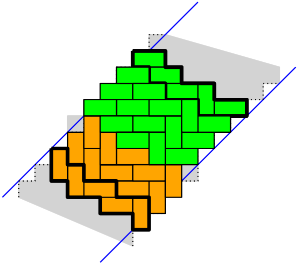
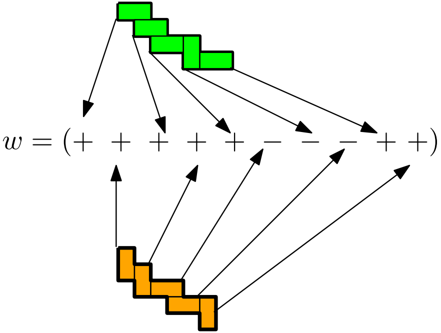
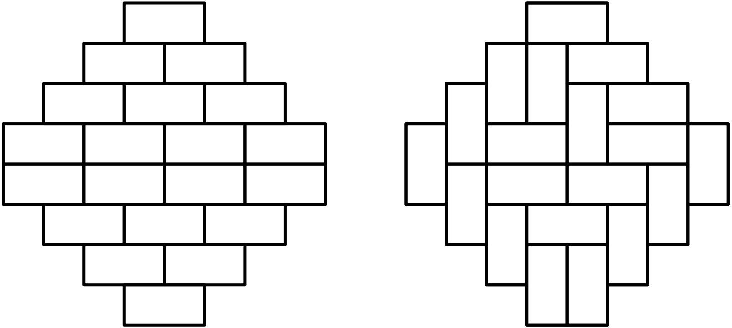
Steep tilings





The steepness condition implies that the tiling is eventually periodic in both directions. The two repeated patterns define the asymptotic data $w \in \{+, -\}^{2\ell}$ of the tiling. For fixed w there is a unique (up to translation) minimal tiling which is periodic from the start.

Examples



Domino tilings of the Aztec diamond [Elkies *et al.*]

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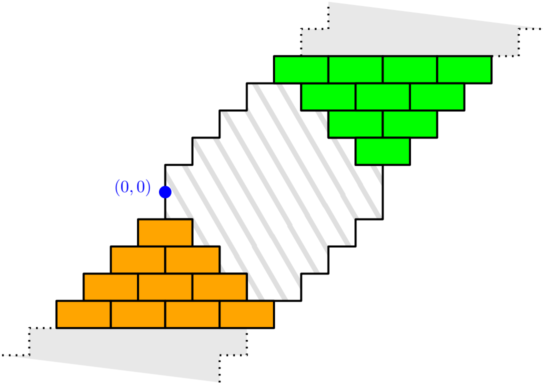
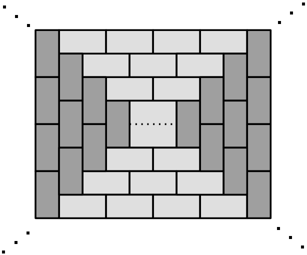
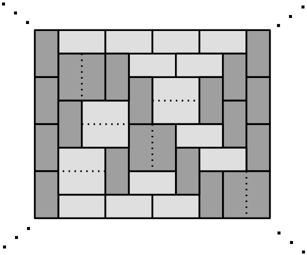


Domino tilings of the Aztec diamond [Elkies *et al.*] correspond to steep tilings with asymptotic data $+-+-+-+-\dots$.

Examples



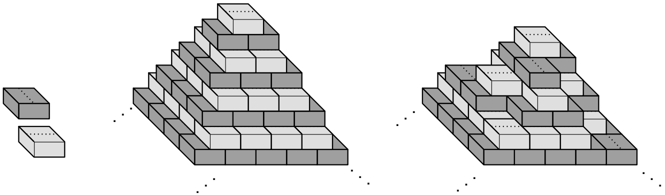
(a)



(b)

Pyramid partitions [Kenyon, Szendrői, Young]

Examples



(a) (b) (c)

Pyramid partitions [Kenyon, Szendrői, Young]

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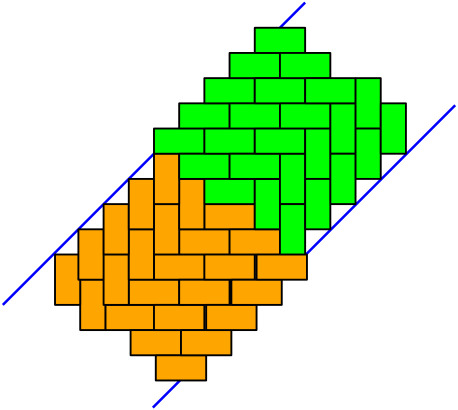

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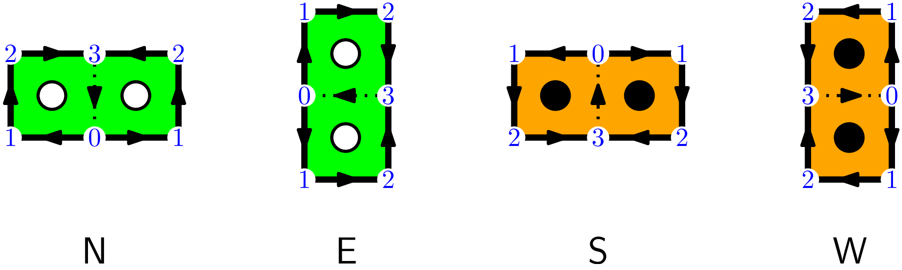
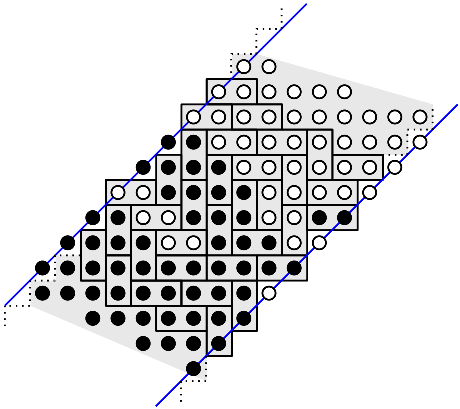
Particle configurations



To each steep tiling we may associate a particle configuration by filling each square covered by a N or E domino with a white particle, and each square covered by a S or W domino with a black particle.

Particle configurations



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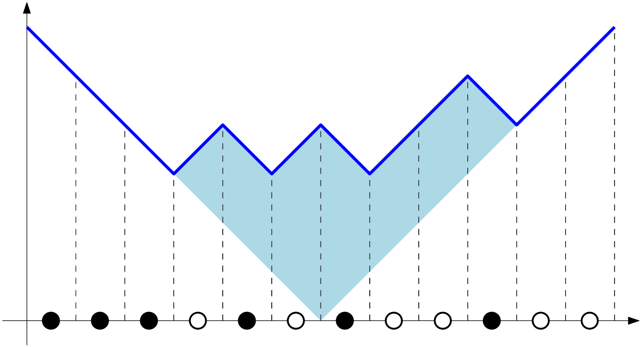
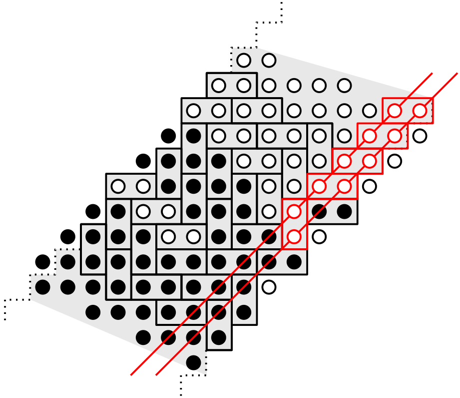
Integer partitions



Particles along a diagonal form a “Maya diagram” which codes an integer partition (here 421).

Interlacing of particles



Between two successive even/odd diagonals, the white particles must be adjacent.

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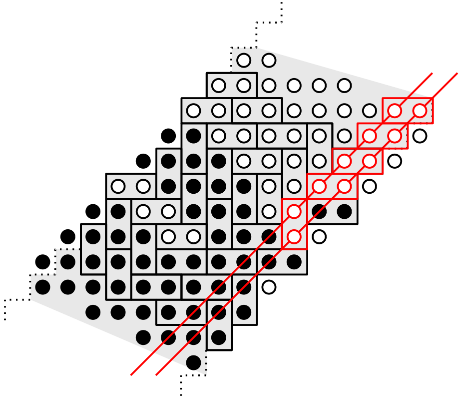
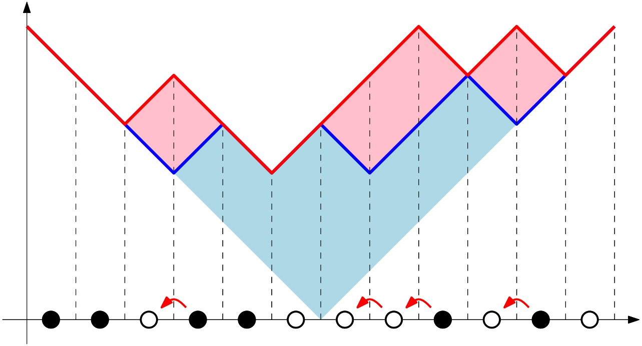


Between two successive even/odd diagonals, the white particles must be adjacent. Conversely, between two successive odd/even diagonals, the black particles must be adjacent.

Interlacing of partitions



Between two successive even/odd diagonals, a finite number of white particles can be moved one site to the left $(+)$ or to the right $(-)$ in the Maya diagram (depending on asymptotic data). This corresponds to adding/removing a horizontal strip to the associated partition.

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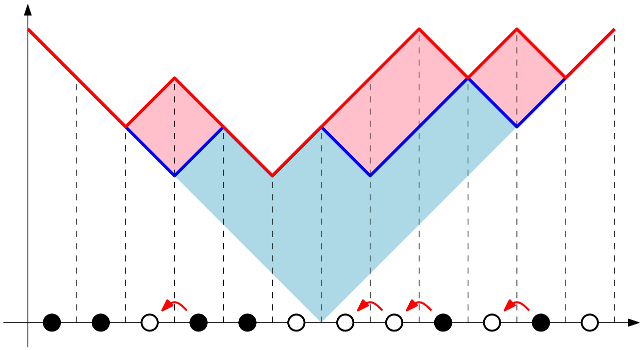

Between two successive even/odd diagonals, a finite number of white particles can be moved one site to the left (+) or to the right (−) in the Maya diagram (depending on asymptotic data). This corresponds to adding/removing a horizontal strip to the associated partition. Conversely, between two successive odd/even diagonals, a vertical strip is added/removed.

Interlacing of partitions

For λ, μ two integer partitions, the following are equivalent:

- λ/μ is a horizontal strip,
- $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \cdots$,
- $\lambda'_i - \mu'_i \in \{0, 1\}$ for all i.

Notation: $\lambda \succ \mu$ or $\mu \prec \lambda$.

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Similarly, the following are equivalent:

- λ/μ is a vertical strip,
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Notation: $\lambda \succ' \mu$ or $\mu \prec' \lambda$.

The fundamental bijection

For a fixed word $w \in \{+,-\}^{2\ell}$, there is a one-to-one correspondence between steep tilings of asymptotic data w and sequences of partitions $(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(2\ell)})$ satisfying for all $k = 1, \ldots, \ell$:

- $\lambda^{(2k-2)} \prec \lambda^{(2k-1)}$ if $w_{2k-1} = +$, and $\lambda^{(2k-2)} \succ \lambda^{(2k-1)}$ if $w_{2k-1} = -$,
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Examples:

- Aztec diamond:
  $\emptyset = \lambda^{(0)} \prec \lambda^{(1)} \succ' \lambda^{(2)} \prec \lambda^{(3)} \succ' \lambda^{(4)} \prec \cdots \succ' \lambda^{(2\ell)} = \emptyset$,
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The size of $\lambda^{(m)}$ is equal to the number of flips on diagonal m in any shortest sequence of flips between the tiling at hand and the minimal tiling.

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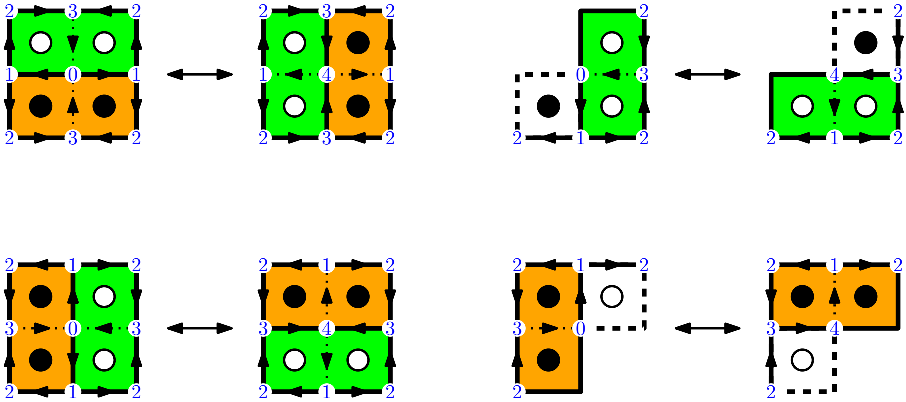
The size of $\lambda^{(m)}$ is equal to the number of flips on diagonal m in any shortest sequence of flips between the tiling at hand and the minimal tiling. Under natural statistics we obtain a Schur process [Okounkov-Reshetikhin].

Flips



2 3 2
1 0 1
2 3 2
2 3 2
1 4 1
2 3 2
2
0 3
2 1 2
2
4 3
2 1 2
2 1 2
3 0 3
2 1 2
2 1 2
3 4 3
2 1 2
2 1 2
3 0
2
2 1 2
3 4
2

Outline

Transfer matrices

Enumerating sequences of interlaced partitions is done via transfer matrices, which are here "vertex operators":

$$\langle\lambda|\Gamma_+(t)|\mu\rangle = \langle\mu|\Gamma_-(t)|\lambda\rangle = \begin{cases} t^{|\mu|-|\lambda|} & \text{if } \lambda \prec \mu \\ 0 & \text{otherwise} \end{cases}$$

$$\langle\lambda|\Gamma'_+(t)|\mu\rangle = \langle\mu|\Gamma'_-(t)|\lambda\rangle = \begin{cases} t^{|\mu|-|\lambda|} & \text{if } \lambda \prec' \mu \\ 0 & \text{otherwise} \end{cases}$$

Example: Aztec diamond:

$$\langle\emptyset|\Gamma_+(z_1)\Gamma'_-(z_2)\Gamma_+(z_3)\Gamma'_-(z_4)\cdots|\emptyset\rangle$$

Bosonic representation

The transfer matrices can be rewritten as

$$\Gamma_{\pm}(t) = \exp \sum_{k \geq 1} \frac{t^k}{k} \alpha_{\pm k}, \qquad \Gamma'_{\pm}(t) = \exp \sum_{k \geq 1} \frac{(-1)^{k-1} t^k}{k} \alpha_{\pm k}$$

where $[\alpha_n, \alpha_m] = n \delta_{n+m}$.

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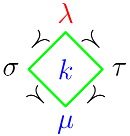
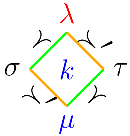
where $[\alpha_n, \alpha_m] = n\delta_{n+m}$. This implies that Γ's with the same sign commute, and that we have the following nontrivial commutation relations:

$$\Gamma_+(t)\Gamma_-(u) = \frac{1}{1 - tu} \Gamma_-(u)\Gamma_+(t)$$

$$\Gamma_+(t)\Gamma'_-(u) = (1 + tu)\Gamma'_-(u)\Gamma_+(t)$$



$k = 0, 1, 2, \ldots$



$k = 0, 1$

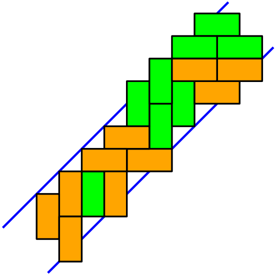
Super Schur functions

When w consists only of $+$'s, the partition function with fixed boundary conditions is a so-called super Schur function

$$\langle \mu | \Gamma_+(x_1)\Gamma'_+(y_1)\Gamma_+(x_2)\Gamma'_+(y_2)\cdots|\lambda\rangle = S_{\lambda/\mu}(x_1,x_2,\ldots;y_1,y_2,\ldots).$$

Super Schur functions may be combinatorially defined in terms of super semistandard tableaux or (reverse) plane overpartitions:

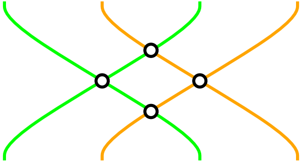
$$\begin{array}{cccc} 1 & 1 & \bar{1} & \bar{2} \\ \bar{1} & 2 & 2 & \bar{2} \\ \bar{1} & \bar{2} & & \\ 2 & & & \end{array}$$



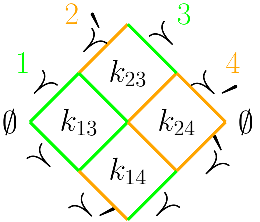
Pure steep tilings

For general asymptotic data and “pure” ($\langle\emptyset|$ and $|\emptyset\rangle$) boundary conditions the partition function is readily evaluated from the commutation relations.

$$\langle\emptyset|\ \Gamma_+(z_1)\ \Gamma'_+(z_2)\ \Gamma_-(z_3)\ \Gamma'_-(z_4)\ |\emptyset\rangle \ =$$



$$\langle\emptyset|\ \Gamma_-(z_3)\ \Gamma'_-(z_4)\ \Gamma_+(z_1)\ \Gamma'_+(z_2)\ |\emptyset\rangle \ \times \ \frac{(1+z_1z_4)(1+z_2z_3)}{(1-z_1z_3)(1-z_2z_4)}$$



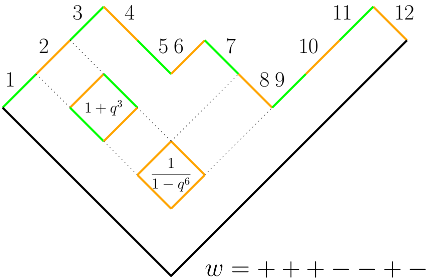
$$k_{13}, k_{24} = 0, 1, 2, \ldots$$
$$k_{23}, k_{14} = 0, 1$$

Equivalently we have a RSK-type bijection between pure steep tilings and suitable fillings of the Young diagram associated with w.

Pure steep tilings

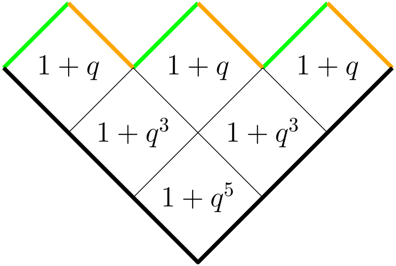
For a general word w and the "q^{flip}" specialization, the partition function of pure steep tilings is given by a hook-length type formula:

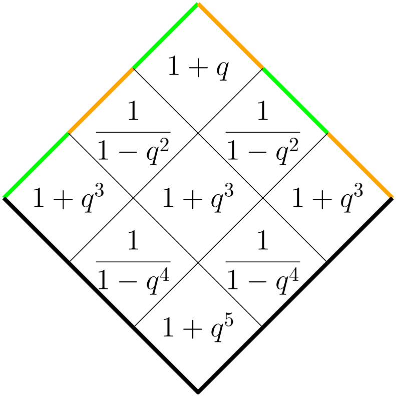
$$T_w(q) = \prod_{\substack{1\leq i<j\leq 2\ell \\ w_i=+,\ w_j=-}} \varphi_{i,j}(q^{j-i}), \qquad \varphi_{i,j}(x) = \begin{cases} 1+x & \text{if } j-i \text{ odd} \\ 1/(1-x) & \text{if } j-i \text{ even} \end{cases}$$



$w = + + + - - + - - + + + -$

Aztec diamonds and pyramids





Aztec diamond $w = + - + - + -$
[Elkies *et al.*, Stanley]

$$T_w(q) = (1+q)^3(1+q^3)^2(1+q^5)$$

Pyramid partitions $w = + + + - - -$.
Case $\ell \to \infty$ [Young]:

$$T_w(q) = \prod_{k \geq 1} \frac{(1+q^{2k-1})^{2k-1}}{(1-q^{2k})^{2k}}$$

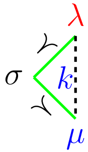
Free boundaries

We may also obtain a closed-form formula for the partition function in the case of free boundary conditions

$$|\underline{v}\rangle = \sum_{\lambda} v^{|\lambda|} |\lambda\rangle$$

thanks to the "reflection relations"

$$\Gamma_+(t)|\underline{v}\rangle = \frac{1}{1-tv}\Gamma_-(tv^2)|\underline{v}\rangle$$
$$\Gamma'_+(t)|\underline{v}\rangle = \frac{1}{1-tv}\Gamma'_-(tv^2)|\underline{v}\rangle$$



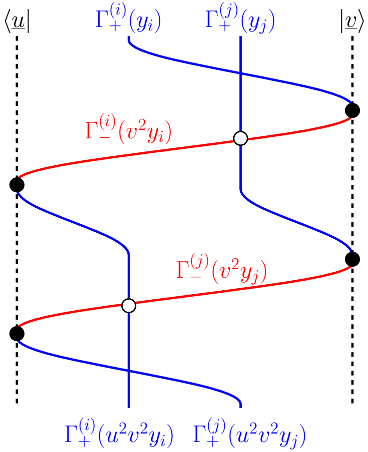
$k = 0, 1, 2, \ldots$

Free boundaries

Example: $w = + + + + \ldots$

$$\langle \underline{u} | \Gamma_+(y_1)\Gamma'_+(y_2)\Gamma_+(y_3)\Gamma'_+(y_4)\cdots|\underline{v}\rangle =$$

$$\prod_{k=1}^{\infty} \left(\frac{1}{1-u^k v^k} \prod_{i=1}^{2\ell} \frac{1}{1-u^{k-1}v^k y_i} \prod_{1\le i<j\le 2\ell} \varphi_{i,j}(u^{2k-2}v^{2k}y_i y_j) \right)$$

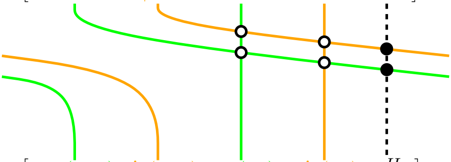


Periodic boundary conditions

When identifying the left and right boundaries we obtain a cylindric steep tiling. The corresponding sequence of interlaced partitions form a periodic Schur process [Borodin].

The partition function may still be written as an infinite product.

$$\mathrm{Tr}[\ \Gamma_+(z_1)\ \Gamma'_+(z_2)\ \Gamma_-(z_3)\ \Gamma'_-(z_4)\ q^H\ ] =$$



$$\mathrm{Tr}[\Gamma_+(qz_1)\Gamma'_+(qz_2)\ \Gamma_-(z_3)\ \Gamma'_-(z_4)\ q^H\ ]\times$$

$$\frac{(1+z_1z_4)(1+z_2z_3)}{(1-z_1z_3)(1-z_2z_4)}$$

Example: $w = + + - -$

$$\mathrm{Tr}\left[\Gamma_+(z_1)\Gamma'_+(z_2)\Gamma_-(z_3)\Gamma'_-(z_4)q^H\right] =$$

$$\prod_{k=1}^{\infty} \frac{(1+q^{k-1}x_1x_4)(1+q^{k-1}x_2x_3)}{(1-q^k)(1-q^{k-1}x_1x_3)(1-q^{k-1}x_2x_4)}$$

Further work

- Correlation functions [joint with C. Boutillier and S. Ramassamy]:
  - straightforward to compute for particles in the pure case, thanks to their free fermionic nature
  - less trivially we deduce an explicit expression for the inverse Kasteleyn matrix, which yields domino correlations
  - more involved in the periodic case [Borodin], how about free boundary case?

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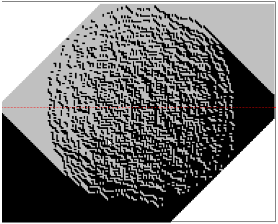
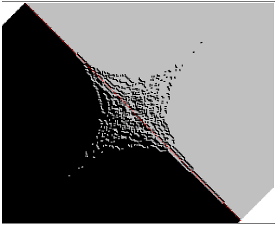
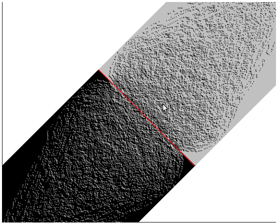
  - more involved in the periodic case [Borodin], how about free boundary case?
- Random generation and limit shapes [joint with D. Betea and M. Vuletić]



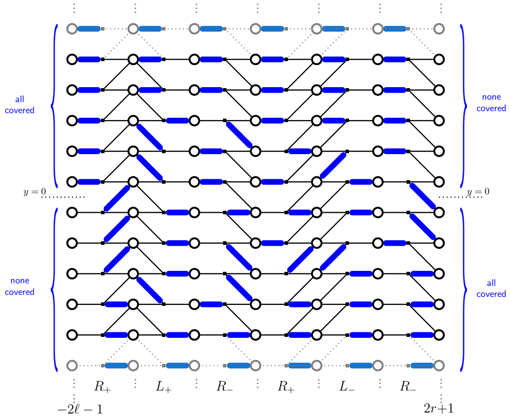




Further work

More general setting [BBCCR]: Rail Yard Graphs (interpolate between lozenge and domino tilings)

- connection with octahedron recurrence/cluster algebras?
- deformations? (e.g. Schur → McDonald)



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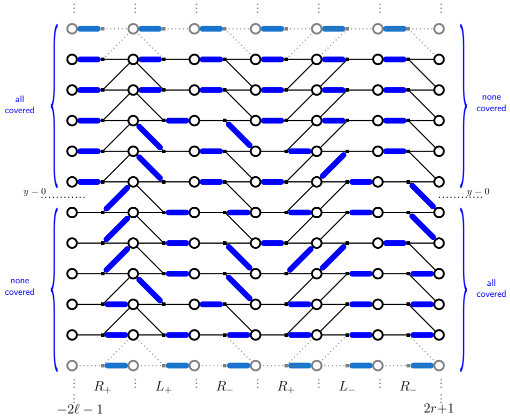

Thanks for your attention!